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NEW SUBCLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS INVOLVING CERTAIN CONVOLUTION OPERATORS

K.O. BABALOLA

Abstract. Let E be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Let A be the class of analytic functions in E, which have the form $f(z) = z + a_2z^2 + \dots$. We define operators $L_n^\sigma \colon A \to A$ using the convolution $*$. Using these operators, we define and study new classes of functions in the unit disk. Moreover, we obtain some basic properties of the new classes, namely inclusion, growth, covering, distortion, closure under certain integral transformation and coefficient inequalities.

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1. INTRODUCTION

Denote by A the class of functions

$$f(z) = z + a_2z^2 + \dots$$

which are analytic in E. Let P be the class of functions

$$p(z) = 1 + c_1z + c_2z^2 + \dots \tag{1}$$

which are also analytic in the unit disk E and satisfy $\operatorname{Re} p(z) > 0$, $z \in E$. Furthermore, for $0 \le \beta < 1$, let $P(\beta)$ denote the subclasses of P consisting of analytic functions of the form $p_\beta(z) = \beta + (1-\beta)p(z)$, $p \in P$.

It is well known that a function $f \in A$ is said to belong to the class $S_0(\beta)$ if $f(z)/z \in P(\beta)$, and is said to be of bounded turning of order β if $f'(z) \in P(\beta)$. The class of functions of bounded turning of order β is denoted by $R(\beta)$ and it is known to consist only of univalent functions in the unit disk. These classes of functions were studied in the literatures [5, 12] and various generalizations of them have appeared in [1, 2, 4, 7].

Let $g(z) = z + b_2z^2 + \dots \in A$. The convolution (or Hadamard product) of f and g (written as $f * g$) is defined as

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Let σ be a fixed real number and $n \in \mathbb{N}$. Define

$$\tau_{\sigma,n}(z) = \frac{z}{(1-z)^{\sigma-(n-1)}}, \quad \sigma - (n-1) > 0$$

and $\tau_{\sigma,n}^{(-1)}$ such that

$$(\tau_{\sigma,n} * \tau_{\sigma,n}^{(-1)})(z) = \frac{z}{1-z}.$$

For $n = 0$, we simply write τ_σ and $\tau_\sigma^{(-1)}$ respectively. Let $f \in A$, define the operator $D^\sigma\colon A \to A$ by

$$D^\sigma f(z) = (\tau_\sigma * f)(z).$$

The operator D^σ is called the Ruscheweyh derivative [9]. Analoguos to D^σ, Noor [6] defined the integral operator $I_\sigma\colon A \to A$ by

$$I_\sigma f(z) = (\tau_\sigma^{(-1)} * f)(z).$$

The operators D^σ and I_σ have been used to define several classes of functions (see [1, 2, 4, 7, 9, 10]). We define the following operators.

Definition 1. Let $f \in A$. We define the operators $L_n^\sigma\colon A \to A$ as follows:

$$L_n^\sigma f(z) = (\tau_\sigma * \tau_{\sigma,n}^{(-1)} * f)(z).$$

Definition 2. Let $f \in A$. We define the operators $l_n^\sigma\colon A \to A$ as follows:

$$l_n^\sigma f(z) = (\tau_\sigma^{(-1)} * \tau_{\sigma,n} * f)(z).$$

Note that $L_0^\sigma f(z) = L_0^0 f(z) = f(z)$, $L_1^1 f(z) = zf'(z)$. Furthermore $L_n^n f(z) = D^n f(z)$ and $L_{-n}^0 f(z) = I_n f(z)$. Similarly, $l_0^\sigma f(z) = l_0^0 f(z) = f(z)$, $l_1^1 f(z) = zf'(z)$, $l_n^n f(z) = I_n f(z)$ and $l_{-n}^0 f(z) = D^n f(z)$. We also have the following remark.

Remark 1. Let $f \in A$. Then

$$L_n^\sigma(l_n^\sigma f(z)) = l_n^\sigma(L_n^\sigma f(z)) = f(z).$$

In the case $\sigma = n$ we write $L_n f(z)$ $(= D^n f(z))$ instead of $L_n^n f(z)$ and $l_n f(z)$ $(= I_n f(z))$ instead of $l_n^n f(z)$.

Next we isolate new classes of functions by:

Definition 3. Let $f \in A$. Let σ be any fixed real number satisfying $\sigma - (n-1) > 0$ for $n \in \mathbb{N}$. Then, for $0 \le \beta < 1$, a function $f \in A$ is said to be in the class $B_n^\sigma(\beta)$ if and only if

$$\operatorname{Re} \frac{L_n^\sigma f(z)}{z} > \beta, \quad z \in E\,. \tag{2}$$

If $\sigma = n$ we write $B_n(\beta)$ in place of $B_n^\sigma(\beta)$. We also note the following equivalent classes of functions: $B_0(\beta) \equiv S_0(\beta)$ and $B_1(\beta) \equiv R(\beta)$. In [4], Goel and Sohi defined classes $M_n(\beta)$ as consisting of functions $f \in A$ satisfying

$$\operatorname{Re} \frac{D^{n+1} f(z)}{z} > \beta, \quad z \in E.$$

These classes coincide with $B^{\sigma}_{n+1}(\beta)$. Following from the geometric condition (2) and Remark 1, functions in the classes $B^{\sigma}_{n}(\beta)$ can be represented in terms of functions in $P(\beta)$ as

$$f(z) = l^{\sigma}_{n}[zp_{\beta}(z)].$$

We investigate the classes $B^{\sigma}_{n}(\beta)$ in Section 3. However, we require some preliminary discussions and results, which we present in the next section.

2. TWO-PARAMETER INTEGRAL ITERATION OF THE CLASS P

In [2], the authors identified the following iterated integral transformation of functions in the class P.

Definition 4 ([2]). Let $p \in P$ and $\alpha > 0$ be real. The *n-th* iterated integral transform of $p(z),\ z \in E$, is defined as

$$p_n(z) = \frac{\alpha}{z^{\alpha}} \int_0^z t^{\alpha-1} p_{n-1}(t)\mathrm{d}t, \quad n \geq 1,$$

with $p_0(z) = p(z)$.

The transformation, denoted by P_n, arose from the study of classes $T^{\alpha}_{n}(\beta)$ consisting of functions defined by the condition $\mathrm{Re}\,\{D^n f(z)^{\alpha}/\alpha^n z^{\alpha}\} > \beta$, where $\alpha > 0$ is real, $0 \leq \beta < 1$, and D^n $(n \in \mathbb{N})$ is the Salagean derivative operator defined as $D^0 f(z) = f(z)$ and $D^n f(z) = z[D^{(n-1)} f(z)]'$ (see [1, 2, 7]); and was applied successfully in providing elegant proofs of many results. It is known that for each $n \geq 1$, the class $T^{\alpha}_{n}(\beta)$ consists only of univalent functions in the unit disk. A basic relationship between the classes P_n and $T^{\alpha}_{n}(\beta)$ was given by the following lemma.

Lemma 1. ([2]) *Let $f \in A$, and α, β and D^n as defined above. Then the following are equivalent:*
(i) $f \in T^{\alpha}_{n}(\beta)$,
(ii) $(D^n f(z)^{\alpha}/\alpha^n z^{\alpha} - \beta)/(1-\beta) \in P$,
(iii) $(f(z)^{\alpha}/z^{\alpha} - \beta)/(1-\beta) \in P_n$.

Analogous to P_n we define the following two-parameter integral iteration of a $p \in P$.

Definition 5. Let $p \in P$. Let σ be any fixed real number such that $\sigma - (n-1) > 0$ for $n \in \mathbb{N}$. We define the *sigma-n-th* integral iteration of $p(z)$, $z \in E$ as

$$p_{\sigma,n}(z) = \frac{\sigma-(n-1)}{z^{\sigma-(n-1)}} \int_0^z t^{\sigma-n} p_{\sigma,n-1}(t)\mathrm{d}t, \quad n \geq 1 \tag{3}$$

with $p_{\sigma,0}(z) = p(z)$.

We note that since $p_{\sigma,0}(z)$ belongs to P, the transform $p_{\sigma,n}(z)$ is analytic, and $p_{\sigma,n}(0)=1$ and $p_{\sigma,n}(z)\neq 0$ for $z\in E$. We denote the family of iterations above by P_n^σ. With $p(z)$ given by (1) it is easily verified that

$$p_{\sigma,n}(z)=1+\sum_{k=1}^{\infty}c_{n,k}^{\sigma}z^k$$

where

$$c_{n,k}^{\sigma}=\frac{\sigma(\sigma-1)\dots(\sigma-(n-1))}{(\sigma+k)(\sigma+k-1)\dots(\sigma+k-(n-1))}c_k,\quad k\geq 1\,. \tag{4}$$

Observe that the multiplier of c_k in (4) can be written in factorial form as:

$$\frac{\sigma(\sigma-1)\dots(\sigma-(n-1))}{(\sigma+k)(\sigma+k-1)\dots(\sigma+k-(n-1))}=\frac{\sigma!}{(\sigma+k)!}\frac{(\sigma+k-n)!}{(\sigma-n)!},\quad k\geq 1.$$

If also, as it is well known, $(\sigma)_n$ stands for the Pochhammer symbol defined by

$$(\sigma)_n=\frac{\Gamma(\sigma+n)}{\Gamma(\sigma)}=\begin{cases}1 & \text{if } n=0,\\ \sigma(\sigma+1)\dots(\sigma+n-1) & \text{if } n\geq 1.\end{cases}$$

then we can write the multiplier as $(\sigma-(n-1))_n/(\sigma+k-(n-1))_n$ and throughout this paper we represent this fraction by $[\sigma]_{n/k}$. Thus we have

$$c_{n,k}^{\sigma}=\frac{(\sigma-(n-1))_n}{(\sigma+k-(n-1))_n}c_k=[\sigma]_{n/k}c_k \tag{5}$$

with $[\sigma]_{0/k}=1$. By setting $p_{\sigma,0}(z)=L_0(z)=(1+z)/(1-z)$ we see easily that the *sigma-n-th* integral iteration of the Mobius functions is

$$L_{\sigma,n}(z)=\frac{\sigma-(n-1)}{z^{\sigma-(n-1)}}\int_0^z t^{\sigma-n}L_{\sigma,n-1}(t)\mathrm{d}t,\quad n\geq 1\,. \tag{6}$$

The function $L_{\sigma,n}(z)$ will play a central role in the family P_n^σ similar to the role of the Mobius function $L_0(z)$ in the family P. Now, from (5) and the fact that $|c_k|\leq 2$ (Caratheodory lemma), we have the following inequality

$$|c_{n,k}^{\sigma}|\leq 2[\sigma]_{n/k},\quad k\geq 1, \tag{7}$$

with equality if and only if $p_{\sigma,n}(z)=L_{\sigma,n}(z)$ given by (6).

Remark 2. From Definitions 4 and 5 we note that $P_1^\sigma=P_1$.

The following results characterizing the family P_n^σ can be obtained *mutatis mutandis* as in Section 2 of [2], thus we omit the proofs.

Theorem 1. *Let $\gamma\neq 1$ be a nonnegative real number. Then for any fixed σ and each $n\geq 1$*

$$\operatorname{Re} p_{\sigma,n-1}(z)>\gamma\Rightarrow \operatorname{Re} p_{\sigma,n}(z)>\gamma,\quad 0\leq\gamma<1,$$

and

$$\operatorname{Re} p_{\sigma,n-1}(z)<\gamma\Rightarrow \operatorname{Re} p_{\sigma,n}(z)<\gamma,\quad \gamma>1.$$

COROLLARY 1. *$P_n^\sigma \subset P$, $n \geq 1$.*

THEOREM 2. *$P_{n+1}^\sigma \subset P_n^\sigma$, $n \geq 1$.*

THEOREM 3. *Let $p_{\sigma,n} \in P_n^\sigma$. Then*
(a) $|p_{\sigma,n}(z)| \leq 1 + 2\sum_{k=1}^{\infty}[\sigma]_{n/k} r^k, \quad |z| = r,$
(b) $\operatorname{Re} p_{\sigma,n}(z) \geq 1 + 2\sum_{k=1}^{\infty}[\sigma]_{n/k}(-r)^k, \quad |z| = r.$
The results are sharp for the function $p_{\sigma,n}(z) = L_{\sigma,n}(z)$ in the upper bound and $p_{\sigma,n}(z) = L_{\sigma,n}(-z)$ in the lower bound.

COROLLARY 2. *$p_{\sigma,n} \in P_n^\sigma$ if and only if $p_{\sigma,n}(z) \prec L_{\sigma,n}(z)$.*

REMARK 3. If we choose $n = 0$ in the corollary above we see that $p \in P$ if and only if $p(z) \prec L_0(z)$ which is well known.

REMARK 4. For $z \in E$, the following are equivalent:
(i) $p \prec L_0(z)$,
(ii) $p \in P$,
(iii) $p_{\sigma,n} \in P_n^\sigma$,
(iv) $p_{\sigma,n}(z) \prec L_{\sigma,n}(z)$.

THEOREM 4. *P_n^σ is a convex set.*

Proof. Let $p_{\sigma,n},\ q_{\sigma,n} \in P_n^\sigma$. Then for nonnegative real numbers μ_1 and μ_2 with $\mu_1 + \mu_2 = 1$, we have

$$\mu_1 p_{\sigma,n} + \mu_2 q_{\sigma,n} = \frac{\sigma - (n-1)}{z^{\sigma-(n-1)}} \int_0^z t^{\sigma-n}(\mu_1 p_{\sigma,n-1} + \mu_2 q_{\sigma,n-1})(t)\mathrm{d}t.$$

The result follows inductively since $\mu_1 p_{\sigma,0} + \mu_2 q_{\sigma,0} = \mu_1 p(z) + \mu_2 q(z) \in P$, for $p,\ q \in P$. $\square$

3. CHARACTERIZATIONS OF THE CLASS $B_N^\sigma(\beta)$

In this section we present the main results of this work. These include inclusion, growth, covering, distortion, closure under certain integral transformation and coefficient inequalities.

First we prove the following lemma, similar to Lemma 1.

LEMMA 2. *Let $f \in A$ and α, β and D^n as defined above. Then the following are equivalent:*
(i) $f \in B_n^\sigma(\beta)$,
(ii) $(L_n^\sigma f(z)/z - \beta)/(1-\beta) \in P$,
(iii) $(f(z)/z - \beta)/(1-\beta) \in P_n^\sigma$.

Proof. That (i) $\Leftrightarrow$ (ii) is clear from Definition 5. Now (ii) is true $\Leftrightarrow$ there exists $p \in P$ such that

$$\begin{aligned} L_n^\sigma f(z) &= z[\beta + (1-\beta)p(z)] \\ &= z + (1-\beta)\sum_{k=1}^{\infty} c_k z^{k+1}\,. \end{aligned} \tag{8}$$

Applying the operator l_n^σ on (8), we have (8) $\Leftrightarrow$

$$f(z) = z + (1-\beta)\sum_{k=1}^{\infty} c_{n,k}^\sigma z^{k+1}$$

$\Leftrightarrow$

$$\frac{f(z)/z-\beta}{1-\beta} = 1 + \sum_{k=1}^{\infty} c_{n,k}^\sigma z^k\,. \tag{9}$$

The right hand side of (9) is a function in P_n^σ. This proves the lemma. □

We prove now the main results.

Theorem 5. *For any fixed σ satisfying $\sigma-(n-1)>0$, the following inclusion holds*

$$B_{n+1}^\sigma(\beta) \subset B_n^\sigma(\beta), \quad n \in \mathbb{N}.$$

Proof. Let $f \in B_{n+1}^\sigma(\beta)$. Then, by Lemma 2, $(f(z)/z-\beta)/(1-\beta) \in P_{n+1}^\sigma$. By Theorem 3, $(f(z)/z-\beta)/(1-\beta) \in P_n^\sigma$. That is, by Lemma 2, again $f \in B_n^\sigma(\beta)$. □

Theorem 6. *The class $B_1^\sigma(\beta)$ consists only of univalent functions in E.*

Proof. Let $f \in B_1^\sigma(\beta)$. Lemma 2 implies that $(f(z)/z-\beta)/(1-\beta) \in P_1^\sigma$. Since σ is any fixed integer satisfying $\sigma-(n-1)>0$, we have $\sigma>0$ for $n=1$ and by Remark 2, it follows that $(f(z)/z-\beta)/(1-\beta) \in P_1$. Thus, by Lemma 1, this implies that the function $f(z)$ belongs to the class $T_1^\sigma(\beta)$ ($\equiv T_1^\alpha(\beta)$) which consists only of univalent functions in E. □

From Theorems 5 and 6 we have

Corollary 3. *For $n \geq 1$, $B_n^\sigma(\beta)$ consists only of univalent functions in E.*

Theorem 7. *Let $f \in B_n^\sigma(\beta)$. Then we have the sharp inequalities*

$$|a_k| \leq 2(1-\beta)[\sigma]_{n/(k-1)}, \quad k \geq 2.$$

Equality is attained for

$$f(z) = z + 2(1-\beta)\sum_{k=2}^{\infty} [\sigma]_{n/(k-1)} z^k\,. \tag{10}$$

Proof. The result follows from equation (9) and the inequality (7). □

Theorem 8. *The class $B_n^\sigma(\beta)$ is closed under the Bernard integral*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\mathrm{d}t, \quad c+1>0\,. \tag{11}$$

Proof. From (11) we have

$$\frac{F(z)/z-\beta}{1-\beta}=\frac{\nu}{z^{\nu}}\int_0^z t^{\nu-1}\left(\frac{f(t)/t-\beta}{1-\beta}\right)\mathrm{d}t, \tag{12}$$

where $\nu=c+1$. Since $f\in B_n^{\sigma}(\beta)$, taking $\nu=c+1=\sigma-n$, we can write (12) as

$$\frac{F(z)/z-\beta}{1-\beta}=\frac{\sigma-n}{z^{\sigma-n}}\int_0^z t^{(\sigma-n)-1}p_{\sigma,n}(t)\mathrm{d}t$$

which implies that $(F(z)/z-\beta)/(1-\beta)\in P_{n+1}^{\sigma}$. Thus, by Theorem 2, we have $(F(z)/z-\beta)/(1-\beta)\in P_n^{\sigma}$. Hence $F\in B_n^{\sigma}(\beta)$. □

THEOREM 9. *Let $f\in B_n^{\sigma}(\beta)$. Then*

$$r+2(1-\beta)\sum_{k=2}^{\infty}(-1)^{k-1}[\sigma]_{n/(k-1)}r^k\le|f(z)|\le r+2(1-\beta)\sum_{k=2}^{\infty}[\sigma]_{n/(k-1)}r^k.$$

The inequalities are sharp.

Proof. The result follows by taking $p_{\sigma,n}(z)=(f(z)/z-\beta)(1-\beta)$ in Theorem 3. Upper bound equality is realized for the function given by (10) while equality in the lower bound equality is attained for the function

$$f(z)=z+2(1-\beta)\sum_{k=2}^{\infty}(-1)k-1[\sigma]_{n/(k-1)}z^k\,. \tag{13}$$

This completes the proof. □

THEOREM 10. *Each function $f(z)$ in the class $B_n^{\sigma}(\beta)$ maps the unit disk onto a domain which covers the disk $|w|<1+2(1-\beta)\sum_{k=2}^{\infty}(-1)k-1[\sigma]_{n/(k-1)}$. The result is sharp.*

Proof. From Theorem 9 it follows that

$$|f(z)|\ge r+2(1-\beta)\sum_{k=2}^{\infty}(-1)^{k-1}[\sigma]_{n/(k-1)}r^k.$$

This implies that the range of every function $f(z)$ in the class $B_n^{\sigma}(\beta)$ covers the disk

$$\begin{aligned}|w|&<1+2(1-\beta)\sum_{k=2}^{\infty}(-1)k-1[\sigma]_{n/(k-1)}\\&=\inf_{r\to1}\left\{r+2(1-\beta)\sum_{k=2}^{\infty}(-1)^{k-1}[\sigma]_{n/(k-1)}r^k\right\}.\end{aligned}$$

The function given by (13) shows that the result is sharp. □

THEOREM 11. *Let $f \in B_n^\sigma(\beta)$. Define*

$$M(\sigma, n, \beta, r) = \sigma - (n-1) + 2(1-\beta)\sum_{k=1}^{\infty}[\sigma]_{(n-1)/k}r^k$$

and

$$m(\sigma, n, \beta, r) = \sigma - (n-1) + 2(1-\beta)\sum_{k=1}^{\infty}[\sigma]_{(n-1)/k}(-r)^k$$

with

$$[\sigma]_{(-1)/k} = \frac{\sigma + k + 1}{\sigma + 1}.$$

Then

$$m(\sigma, n, \beta, r) \leq \left|(\sigma - n)\frac{f(z)}{z} + f'(z)\right| \leq M(\sigma, n, \beta, r).$$

The inequalities are sharp.

Proof. Since $f \in B_n^\sigma(\beta)$, by Lemma 2, there exists $p_{\sigma,n} \in P_n^\sigma$ such that

$$f(z) = z[\beta + (1-\beta)p_{\sigma,n}(z)]\,. \tag{14}$$

Hence we have

$$f'(z) = \beta + (1-\beta)[p_{\sigma,n}(z) + zp'_{\sigma,n}(z)]\,. \tag{15}$$

From (14) and (15) we get

$$(\sigma - n)\frac{f(z)}{z} + f'(z) = (\sigma-(n-1))\beta + (1-\beta)[(\sigma-(n-1))p_{\sigma,n} + zp'_{\sigma,n}]\,. \tag{16}$$

However we find from (3) that

$$(\sigma - (n-1))p_{\sigma,n}(z) + zp'_{\sigma,n}(z) = (\sigma - (n-1))p_{\sigma,n-1}(z)$$

so that (16) becomes

$$(\sigma - n)\frac{f(z)}{z} + f'(z) = (\sigma - (n-1))[\beta + (1-\beta)p_{\sigma,n-1}].$$

Therefore, by Theorem 3, we get

$$\left|(\sigma - n)\frac{f(z)}{z} + f'(z)\right| \leq \sigma - (n-1) + 2(1-\beta)\sum_{k=1}^{\infty}[\sigma]_{(n-1)/k}r^k \tag{17}$$

and

$$\mathrm{Re}\left\{(\sigma - n)\frac{f(z)}{z} + f'(z)\right\} \geq \sigma - (n-1) + 2(1-\beta)\sum_{k=1}^{\infty}[\sigma]_{(n-1)/k}(-r)^k\,. \tag{18}$$

The inequalities now follow from (17) and (18). Upper bound equality is realized for the function $f(z)$ given by (10) while equality in the lower bound equality is attained for the function $f(z)$ defined by (13). □

Finally we prove

THEOREM 12. *$B_n^\sigma(\beta)$ is a convex family of analytic and univalent functions.*

Proof. Let $f,\ g \in B_n^\sigma(\beta)$. Then by Lemma 2 there exists $p_{\sigma,n},\ q_{\sigma,n} \in P_n^\sigma$ such that

$$f(z) = z[\beta + (1-\beta)p_{\sigma,n}(z)]$$

and

$$g(z) = z[\beta + (1-\beta)q_{\sigma,n}(z)].$$

Therefore for nonnegative real numbers μ_1 and μ_2 with $\mu_1 + \mu_2 = 1$, we have

$$\begin{aligned} h(z) = \mu_1 f(z) + \mu_2 g(z) &= z\mu_1[\beta + (1-\beta)p_{\sigma,n}(z)] + z\mu_2[\beta + (1-\beta)q_{\sigma,n}(z)] \\ &= z[(\mu_1+\mu_2)\beta + (1-\beta)(\mu_1 p_{\sigma,n} + \mu_2 q_{\sigma,n}] \\ &= z[\beta + (1-\beta)(\mu_1 p_{\sigma,n} + \mu_2 q_{\sigma,n}]. \end{aligned}$$

The conclusion follows from Theorem 4. □

4. GENERAL REMARKS

The two-parameter integral iteration of the Caratheodory functions presented in Section 2 of this paper has also proved very resourceful in providing elegantly short proofs of many fundamental results in the theory of analytic and univalent functions. An earlier one presented in [2] closely relates with certain classes of functions defined by the Salagean derivative operators. This provides the motivation to search for analogous iteration that will equivalently closely relate with certain other classes of functions involving the Ruscheweyh derivative, and this leads us to defining new operators $L_n^\sigma\colon A \to A$, which includes the Ruscheweyh derivative as a special case.

Finally we remark that the results presented in this work include many earlier ones as particular cases.

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Department of Mathematics
University of Ilorin
Ilorin, Nigeria
E-mail: abuuabdilqayyuum@gmail.com