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GROUPS OF INFINITE UNITRIANGULAR MATRICES

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Abstract. In this paper we define the group of unitriangular matrices over any well-ordered set and we study their properties. We prove that for any ordinal α there is a nilpotent group having its nilpotency class exactly α.

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1. INTRODUCTION

Let R be an associative ring with unity and $n \geq 1$ a natural number. We denote, as usually, by $\mathrm{M}_n(R)$ the set of all $n \times n$ matrices over R, by $\mathrm{NT}_n(R)$ the set of $n \times n$ matrices over R which have only zero elements below the main diagonal and by $\mathrm{UT}_n(R)$ we denote the set $\mathrm{I}_{n,R} + \mathrm{NT}_n(R)$ (where $\mathrm{I}_{n,R}$ is the $n \times n$ unity matrix over R). The last two sets are called the nilpotent group of order n over R, respectively the unitriangular group of order n over R (one can easily check that these are indeed groups with respect to the usual multiplication of matrices). For every two natural numbers $1 \leq i, j \leq n$ we denote by $e_{ij}(a)$ the matrix in $\mathrm{M}_n(R)$ that has only zero entries, except for the intersection of the i^{th} row and the j^{th} column, where it has an a, and by $t_{ij}(a)$ we denote the matrix $\mathrm{I}_{n,R} + e_{ij}(a)$. The matrices e_{ij} are called elementary matrices, whereas the t_{ij} matrices are called transvections (we omitted the n and the R from the above notations in order to improve the readability of the text).

Furthermore, for every natural number $1 \leq m \leq n$ we denote by $\mathrm{UT}_n^m(R)$ the set of matrices from $\mathrm{UT}_n(R)$ whose first $m-1$ diagonals above the first diagonal contain only zeroes, hence for every $1 \leq m \leq n$ we have

$$\mathrm{UT}_n^m(R) = \{M \in \mathrm{UT}_n(R) \mid \forall 1 \leq i < j \leq n,\ i < j + m : M(i,j) = 0\}.$$

We notice that the group $\mathrm{UT}_n(R)$ is generated by the family of transvections t_{ij} with $i \leq j$ and for every natural number $1 \leq m \leq n$ the set $\mathrm{UT}_n^m(R)$ is generated by the family of transvections t_{ij} with $j \geq i+m$. From this it follows immediately that the sets $\mathrm{UT}_n^m(R),\ 1 \leq m \leq n$ are subgroups of $(\mathrm{UT}_n(R), \cdot)$. These groups have been investigated by A.J. Weir [8] and V.M. Levchuk [2], [3], [4] and [5]. It can be proved (see [1, p. 35, 128–130]) that

$$\mathrm{UT}_n(R) = \mathrm{UT}_n^1(R) \trianglerighteq \cdots \trianglerighteq \mathrm{UT}_n^n(R) = \mathrm{I}_n,$$

so the group $\mathrm{UT}_n(R)$ is nilpotent, and its nilpotency class is exactly $n-1$. The lower and the upper central series coincide (see the same reference) and they consist of the sets $\mathrm{UT}_n^i(R)$, where $i = 1, \ldots, n$. We conclude that for every natural number n there exists a nilpotent group having the nilpotency class exactly n, namely $\mathrm{UT}_{n+1}(R)$.

Now we proceed to generalize the above notions and results to the case of matrices indexed by well-ordered sets.

2. THE UNITRIANGULAR GROUP FOR INFINITE INDEX SETS

Let $(R, +, \cdot)$ be an associative ring with unity and $(A, \leq)$ a well-ordered set, whose ordinal will be denoted by α.

Definition 2.1. The set

$$\mathrm{M}_A(R) = \{M : A \times A \to R \mid |\{(i, j) \in A \times A \mid i \neq j \text{ and } M(i, j) \neq 0\}| < \infty\}$$

is called the set of all $\alpha \times \alpha$ matrices over R.

Remark 2.2. From the definition above it follows that the set $\mathrm{M}_A(R)$ consists of all the $\alpha \times \alpha$ matrices over R that have a finite number of nonzero elements except the first diagonal. This restriction was caused by the impossibility of introducing certain notions in less restrictive conditions.

We define the addition as follows. For $M,\ N \in \mathrm{M}_A(R)$ define $P = M + N \in \mathrm{M}_A(R)$ by

$$P(i, j) = M(i, j) + N(i, j),$$

for all $i, j \in A$. For the same two elements, we define their product $Q \in \mathrm{M}_A(R)$ by

$$Q(i, j) = \sum_{k \in A} M(i, k)N(k, j)$$

for all $i,\ j \in A$, or, equivalently,

$$Q(i, j) = \sum_{k < \alpha} M(i, k)N(k, j),$$

for all $i,\ j \in A$, since $A = \{k \mid k < \alpha\}$. Note that the above sums are finite.

It is clear that $(\mathrm{M}_A(R), +, \cdot)$ is an associative ring with unity.

Definition 2.3. The set

$$\mathrm{NT}_A(R) = \{M \in \mathrm{M}_A(R) \mid \forall i, j \in A, j \leq\ i : M(i, j) = 0\}$$

is called the set of $\alpha \times \alpha$ superior triangular matrices over R.

Remark 2.4. $\mathrm{NT}_A(R)$ consists of all the $\alpha \times \alpha$ matrices on R that have only zeroes on and below the first diagonal and a finite number of nonzero entries above it.

Lemma 2.5. *The set* $\mathrm{NT}_A(R)$ *is a ring with the induced operations, and it is a subring of* $\mathrm{M}_A(R)$.

Proof. We notice that $\mathrm{I}_{A,R} \in \mathrm{NT}_A(R)$. Let M and N be two elements of $\mathrm{NT}_A(R)$. We prove that $M - N \in \mathrm{NT}_A(R)$ and $MN \in \mathrm{NT}_A(R)$. Indeed, for every $i, j \in A, j \leq i$ we have

$$(M - N)(i,j) = M(i,j) - N(i,j) = 0,$$

and

$$(MN)(i,j) = \sum_{k \in A} M(i,k)N(k,j) = 0,$$

because if $k \leq i$ then $M(i,k) = 0$ and if $i < k$ then $j < k$, so $N(k,j) = 0$. $\square$

Remark 2.6. $\mathrm{NT}_A(R)$ is neither a right nor a left ideal of the ring $\mathrm{M}_A(R)$. For example $e_A^{21}(R) \in \mathrm{M}_A(R)$ and $e_A^{12}(R) \in \mathrm{NT}_A(R)$, where $\alpha \geq 2$, but $e_A^{21}(R) \cdot e_A^{12}(R)$ has a 1 at the intersection of the second row and the second column, hence it is not an element of $\mathrm{NT}_A(R)$.

Definition 2.7. The set $\mathrm{UT}_A(R) = \mathrm{I}_{A,R} + \mathrm{NT}_A(R)$ is called the set of $\alpha \times \alpha$ unitriangular matrices over R.

Remark 2.8. The set $\mathrm{UT}_A(R)$ consists of all the $\alpha \times \alpha$ matrices over R that have zero entries below the first diagonal, 1 on the first diagonal and a finite number of nonzero entries above it.

Theorem 2.9. *The set* $\mathrm{UT}_A(R)$ *forms a group with respect to the multiplication of the infinite matrices.*

Proof. We notice that $\mathrm{I}_{A,R} \in \mathrm{UT}_A(R)$. Let P and Q be two elements in $\mathrm{UT}_A(R)$. Then we can find two matrices M and N in $\mathrm{M}_A(R)$ such that $P = 1 + M$ and $Q = 1 + N$. Then we can write

$$PQ = 1 + (M + N + MN),$$

where $M + N + MN \in \mathrm{UT}_A(R)$, so $PQ \in \mathrm{UT}_A(R)$. Hence it suffices to show that every matrix in $\mathrm{UT}_A(R)$ is invertible and its inverse belongs to $\mathrm{UT}_A(R)$. Now let P be an element of $\mathrm{UT}_A(R)$ and it is straightforward to prove that there exists a matrix Q in $\mathrm{UT}_A(R)$ such that $PQ = \mathrm{I}_{A,R} = QP$ (actually we prove that the infinite system of linear equations that follow from the equality of the entries of the matrices PQ and $\mathrm{I}_{A,R}$ has a unique solution). Or, likewise in the finite case, we can define the determinant of an $\alpha \times \alpha$ matrix as a multilinear alternating operator, and consequently, the algebraic complement of its entries. Clearly, $\det(P) = 1$, so P is invertible. For every $i,\ j \in A, j < i$ we notice that the algebraic complement of $P(i,j)$ is zero, hence $P^{-1}(i,j) = 0$ and if $i = j$, then the algebraic complement of $P(i,j)$ is 1, hence $P^{-1}(i,j) = 1$. Thus we deduce that the inverse of P belongs to $\mathrm{UT}_A(R)$. $\square$

Remark 2.10. $\mathrm{UT}_A(R)$ is not a subgroup of $(\mathrm{M}_A(R), +)$. Indeed, for example $\mathrm{I}_{A,R} \in \mathrm{UT}_A(R)$ and $\mathrm{I}_{A,R} + e_A^{21}(R) \in \mathrm{UT}_A(R)$, but

$$\mathrm{I}_{A,R} + e_A^{21}(R) - \mathrm{I}_{A,R} = e_A^{21}(R)$$

does not belong to $\mathrm{UT}_A(R)$.

THEOREM 2.11. *The set* $\mathrm{UT}_A(R)$ *is the subgroup of* $(\mathrm{M}_A(R), \cdot)$ *generated by the transvections* $t_{ij}(a)$ *with* $i < j$ *and* $a \in R, a \neq 0$.

Proof. It can be easily checked that multiplying the matrix $M \in \mathrm{UT}_A(R)$ on the right with the transvection $t_{ij}(a), i < j, a \in R, a \neq 0$ we add the i^{th} column multiplied with a to the j^{th} column, whereas multiplying the same matrix on the left by the transvection $t_{ij}(a)$, $i < j$, $a \in R$, $a \neq 0$, we add the j^{th} row multiplied with a to the i^{th} row (here by the "i^{th}" column we mean the $\{M(i,\beta) \mid \beta \in A\}$ set and by the "i^{th}" column we mean the set $\{M(\beta, i) \mid \beta \in A\}$.

The product of two transvections $t_{ij}(a)$ and $t_{kl}(b)$ with $i < j, k < l$ and $a, b \in R$ is clearly in $\mathrm{UT}_A(R)$ (because adding a row multiplied with an element in R to a previous row or performing a similar operation on the columns of a matrix from $\mathrm{UT}_A(R)$ we obtain a element from $\mathrm{UT}_A(R)$). Hence the subgroup of $\mathrm{M}_A(R)$ generated by the transvections $t_{ij}(a)$ with $i < j$ and $a \in R$ is included in the set $\mathrm{UT}_A(R)$ (we also have to notice that the inverse of the transvection $t_{ij}(a)$ is the transvection $t_{ij}(-a)$).

Now we prove that the set $\mathrm{UT}_A(R)$ is included in the subgroup of $\mathrm{M}_A(R)$ generated by the transvections $t_{ij}(a)$ with $i < j$ and $a \in R$, in other words that every element of the set $\mathrm{UT}_A(R)$ can be written as a finite product of such transvections. But since every element of the above set has a finite number of nonzero elements above the first diagonal, after multiplying with a finite number of transvections either on the right or on the left we transform the initial matrix to the unity matrix. Then the inverses of the above transvections (in reverse order) yield a decomposition of the initial matrix in transvections with the required property. □

DEFINITION 2.12. For every ordinal $1 \leq \beta \leq \alpha$ we denote by $\mathrm{UT}_A^\beta(R)$ the set

$$\{M \in \mathrm{UT}_A(R) \mid \forall i, j \in A, i < j < i + \beta : M(i,j) = 0\}$$

and for every limit ordinal $1 \leq \beta \leq \alpha$ we define

$$\mathrm{UT}_A^\beta(R) = \bigcap_{\gamma < \beta} \mathrm{UT}_A^\gamma(R).$$

REMARK 2.13. The set $\mathrm{UT}_A^\beta(R)$ consists of the $\alpha \times \alpha$ matrices over R with the property that all their diagonals above the first diagonal and at the left of the "β diagonal" are completely zero (by the "β diagonal" we mean the set $\{(i, i + \beta) \mid i \in A\}$). If β is not a limit ordinal, then we can say that the set $\mathrm{UT}_A^\beta(R)$ consists of the $\alpha \times \alpha$ matrices over R with the property that their first $\beta - 1$ diagonals above the first diagonal are completely zero, whereas if β is a limit ordinal the last sentence does not make sense.

THEOREM 2.14. *We have that* $\mathrm{UT}_A^1(R) = \mathrm{UT}_A(R)$ *and* $\mathrm{UT}_A^\alpha = \mathrm{I}_{A,R}$. *In addition,* $\mathrm{UT}_A^\beta(R) = \mathrm{UT}_A(R)$ *if and only if* $\beta = 1$, *and* $\mathrm{UT}_A^\beta = \mathrm{I}_{A,R}$ *if and only if* $\beta = \alpha$.

Proof. Clearly,

$$\begin{aligned}\mathrm{UT}^1_A(R) &= \{M \in \mathrm{UT}_A(R) \mid \forall i, j \in A, i < j < i+1 : M(i,j) = 0\}\\ &= \mathrm{UT}_A(R),\end{aligned}$$

because the condition $i < j < i+1$ is not fulfilled by any ordinals $i, j \in A$. Now suppose that $\mathrm{UT}^\beta_A(R) = \mathrm{UT}_A(R)$ and we prove that $\beta = 1$.

If β is not a limit ordinal, then we consider the matrix $M \in \mathrm{UT}_A(R)$ with the property that for every $i \in A$, $M(i, i{+}1) = 1$. Then M belongs to $\mathrm{UT}^\beta_A(R)$. If we suppose that $\beta > 1$, then we have $i < i+1 < i+\beta$, but $M(i, i+1) \neq 0$, which contradicts the definition of the set $\mathrm{UT}^\beta_A(R)$ and therefore we deduce that $\beta = 1$.

If β is a limit ordinal, then from the equality

$$\bigcap_{\gamma<\beta} \mathrm{UT}^\beta_A(R) = \mathrm{UT}_A(R)$$

we obtain that for every $\gamma < \beta$, $\mathrm{UT}^\gamma_A(R) = \mathrm{UT}_A(R)$. Since β is a limit ordinal, we have $\beta > 2$, so there exists an ordinal $\gamma < \beta$ such that $\gamma > 1$, for which we have already seen that $\mathrm{UT}^\gamma_A \neq \mathrm{UT}_A(R)$. Hence in this case the equality $\mathrm{UT}^\beta_A(R) = \mathrm{UT}_A(R)$ is impossible.

We prove that $\mathrm{UT}^\alpha_A(R) = \mathrm{I}_{A,R}$. If α is not a limit ordinal, then UT^α_A consists of the $\alpha \times \alpha$ matrices over R with the property that for every $i, j \in A, i < j < i + \alpha$, we have $M(i,j) = 0$. But the condition $i < j < i + \alpha$ is satisfied by every ordinals $i, j \in A, i < j$ (since $i, j < \alpha$), so indeed $\mathrm{UT}^\alpha_A(R) = \mathrm{I}_{A,R}$.

If α is a limit ordinal, then

$$\mathrm{UT}^\alpha_A = \bigcap_{\beta<\alpha} \mathrm{UT}^\beta_A(R).$$

It is clear that $\mathrm{I}_{A,R} \in \mathrm{UT}^\alpha_A(R)$. Consider an element M of the set $\mathrm{UT}^\alpha_A(R)$ and $i, j \in A, i < j$. Then $i < j < i + j$ and $M \in \mathrm{UT}^j_A(R)$ because $\mathrm{UT}^\alpha_A(R) \subseteq \mathrm{UT}^j_A(R)$, from where we deduce that $M(i,j) = 0$. Therefore $\mathrm{UT}^\alpha_A(R) = \mathrm{I}_{A,R}$.

Now let us suppose that $\mathrm{UT}^\beta_A = \mathrm{I}_{A,R}$ and $\beta < \alpha$. If β is not a limit ordinal, consider the matrix $M \in \mathrm{UT}_A(R)$ with the property that for every $i \in A$, $M(i, i + \beta + 1) = 0$, and which is zero elsewhere. Then for every $i, j \in A$, $i < j < i + \beta$, we have that $M(i,j) = 0$. Hence according to the definition of the set $\mathrm{UT}^\beta_A(R)$, we have that $M \in \mathrm{UT}^\beta_A(R)$. It follows that $M = \mathrm{I}_{A,R}$, which is a contradiction and therefore $\beta = \alpha$.

If β is a limit ordinal, then consider the matrix $M \in \mathrm{UT}_A(R)$ with the property that for every $i \in A$, $M(i, i{+}1) = 1$, and which is zero elsewhere. We have that for every $\gamma < \beta$, $M \in \mathrm{UT}^\gamma_A(R)$, therefore $M \in \mathrm{UT}^\beta_A(R)$. But $M \neq \mathrm{I}_{A,R}$, a contradiction. So in this case $\mathrm{UT}^\beta_A(R)$ cannot be equal to $\mathrm{I}_{A,R}$. □

LEMMA 2.15. *For every non-limit ordinal $\beta \leq \alpha$ the set $\mathrm{UT}_A^\beta(R)$ is the subgroup of $\mathrm{UT}_A(R)$ generated by the transvections $t_{ij}(a)$, where $i, j \in A$, $j \geq i+\beta$, $a \in R$, $a \neq 0$.*

Proof. It can be easily checked that the product of two such transvections belongs to $\mathrm{UT}_A^\beta(R)$, so the subgroup of $\mathrm{UT}_A(R)$ generated by the transvections $t_{ij}(a)$, where $i, j \in A$, $j \geq i+\beta$, $a \in R$, $a \neq 0$ is included in $\mathrm{UT}_A^\beta(R)$.

As above, we prove that every element of the set $\mathrm{UT}_A^\beta(R)$ can be written as a product of a finite number of transvections with the required property (it is esential that every element of $\mathrm{UT}_A^\beta(R)$ has a finite number of nonzero elements above the first diagonal). □

THEOREM 2.16. *For every $\beta \leq \alpha$, $\mathrm{UT}_A^\beta(R)$ is a subgroup of $\mathrm{UT}_A(R)$.*

Proof. If β is not a limit ordinal this proposition follows immediately from the previous one. However, we present here an alternative proof of the inclusion

$$\mathrm{UT}_A^\beta(R) \cdot \mathrm{UT}_A^\beta(R) \subseteq \mathrm{UT}_A^\beta(R)$$

for every non-limit ordinal β. For $\beta = 1$ the conclusion is immediate. We suppose now that $\beta \geq 2$ and let M, N be two elements of $\mathrm{UT}_A^\beta(R)$. Then clearly $M, N \in \mathrm{UT}_A(R)$, hence we also have have that $MN \in \mathrm{UT}_A(R)$. It suffices to show that for every $i, j \in A, i < j < i+\beta$ we have $(MN)(i,j) = 0$. Let i, j be ordinals in A such that $i < j < i+\beta$. For $k < i$ we have $M(i,k) = 0$ (because $M \in \mathrm{UT}_A(R)$), for $i < k < i+\beta$ we also have $M(i,k) = 0$ (because $M \in \mathrm{UT}_A^\beta(R)$) and, finally, for $k > j$ we obtain $N(k,j) = 0$ from $N \in \mathrm{UT}_A(R)$. Hence

$$(MN)(i,j) = \sum_{k \in A} M(i,k)N(k,j) = N(i,j) = 0$$

since $N \in \mathrm{UT}_A^\beta(R)$.

Now let us consider the case in which β is a limit ordinal. If $\beta = \omega$, then

$$\mathrm{UT}_A^\omega(R) = \bigcap_{n \in (N)} \mathrm{UT}_n(R)$$

is a subgroup of $\mathrm{UT}_A(R)$ (being an intersection of subgroups). We suppose that $\mathrm{UT}_A^\gamma(R)$ is a subgroup of the group $\mathrm{UT}_A(R)$ for every limit ordinal $\gamma < \beta$. Then

$$\mathrm{UT}_A^\beta(R) = (\bigcap_{\gamma \in \mathrm{I}_1} \mathrm{UT}_A^\gamma(R)) \cap (\bigcap_{\gamma \in \mathrm{I}_2} \mathrm{UT}_A^\gamma(R)),$$

where I_1 denotes the set of all non-limit ordinals smaller than β and I_2 denotes the set of all limit ordinals smaller than β. Using the inductive hypothesis and the conclusion of the theorem already proven for non-limit ordinals, we deduce that $\mathrm{UT}_A^\beta(R)$ is a subgroup of $\mathrm{UT}_A(R)$, being an intersection of subgroups. □

3. THE NILPOTENCY OF THE UNITRIANGULAR GROUP

THEOREM 3.1. *For every ordinal $\beta < \alpha$ the following equality holds:*

$$[\mathrm{UT}_A(R), \mathrm{UT}_A^{\beta}(R)] = \mathrm{UT}_A^{\beta+1}(R).$$

Proof. If β is not a limit ordinal, then it is enough to use the fact that $\mathrm{UT}_A^{\beta}(R)$ is generated by the transvections $t_{ij}(a)$, where $i, j \in A$, $j \geq i + \beta$, $a \in R$, $a \neq 0$, and that

$$[t_{ik}(a), t_{kj}(b)] = t_{ij}(ab).$$

for all $i, j, k \in A$ and $a, b \in R$. Indeed, $[\mathrm{UT}_A(R), \mathrm{UT}_A^{\beta}(R)]$ consists of finite products of commutators $[x, y]$, where $x \in \mathrm{UT}_A(R)$ and $y \in \mathrm{UT}_A^{\beta}(R)$. But both x can be written as finite products of transvections $t_{ij}(a)$, where $i, j \in A$, $j > i$, $a \in T$, $a \neq 0$, and y can be written as finite products of transvections $t_{ij}(a)$, where $i, j \in A, j \geq i + \beta, a \in R, a \neq 0$, so $[x, y]$ can be written as a finite product of transvections $t_{ij}(a)$, where $i, j \in A$, $j \geq i + \beta + 1$, $a \in R$, $a \neq 0$. Therefore all the elements of the set $[\mathrm{UT}_A(R), \mathrm{UT}_A^{\beta}(R)]$ will belong to $\mathrm{UT}_A^{\beta+1}(R)$.

Conversely, every element of the set $\mathrm{UT}_A^{\beta+1}(R)$ can be written as a product of transvections $t_{ij}(a)$, where $i, j \in A, j \geq i + \beta + 1, a \in R, a \neq 0$. Then each such transvection can be written as the commutator of two other transvections (by using the above formula). It follows that the product of the obtained commutators be an element of $[\mathrm{UT}_A(R), \mathrm{UT}_A^{\beta}(R)]$.

Now let β be a limit ordinal. We suppose that the conclusion holds for every limit ordinal smaller than β. Then

$$\begin{aligned}
[\mathrm{UT}_A(R), \mathrm{UT}_A^{\beta}(R)] &= \langle [x, y] \mid x \in \mathrm{UT}_A(R), y \in \mathrm{UT}_A^{\beta}(R) \rangle \\
&= \langle [x, y] \mid x \in \mathrm{UT}_A(R), y \in \bigcap_{\gamma<\beta} \mathrm{UT}_A^{\gamma}(R) \rangle \\
&= \langle \bigcap_{\gamma<\beta} [x, y] \mid x \in \mathrm{UT}_A(R), y \in \mathrm{UT}_A^{\gamma}(R) \rangle \\
&= \bigcap_{\gamma<\beta} \langle [x, y] \mid x \in \mathrm{UT}_A(R), y \in \mathrm{UT}_A^{\gamma}(R) \rangle \\
&= \bigcap_{\gamma<\beta} [\mathrm{UT}_A(R), \mathrm{UT}_A^{\gamma}(R)] \\
&= \bigcap_{\gamma<\beta} \mathrm{UT}_A^{\gamma+1}(R) = \mathrm{UT}_A^{\beta+1}(R),
\end{aligned}$$

so the conclusion holds for every ordinal β. □

REMARK 3.2. From the theorem above it follows that the lower central series of the group $\mathrm{UT}_A^{\alpha}(R)$ consists of the sets $\mathrm{UT}_A^{\beta}(R)$, where $\beta \leq \alpha$.

Analogously, the group $\mathrm{UT}_A^\alpha(R)$ is nilpotent because its upper central series is

$$\mathrm{UT}_A(R) = \mathrm{UT}_A^1(R) \trianglerighteq \cdots \trianglerighteq \mathrm{UT}_A^\alpha(R) = \mathrm{I}_{A,R}.$$

It is sufficient to prove that the subgroups $\mathrm{UT}_A^\beta(R)$ are normal and that the above series is central, in other words that for each ordinal $\beta < \alpha$ we have

$$\mathrm{UT}_A^\beta(R)/\mathrm{UT}_A^{\beta+1}(R) \leq Z(\mathrm{UT}_A(R)/\mathrm{UT}_A^{\beta+1}(R)).$$

Theorem 3.3. *For every ordinal β the subgroup $\mathrm{UT}_A^\beta(R)$ is normal in $\mathrm{UT}_A(R)$.*

Proof. Suppose that β is not a limit ordinal. Than we have to prove that for every $N \in \mathrm{UT}_A(R)$, for every $M \in \mathrm{UT}_A^\beta(R)$ and for every $i, j \in A$, $i < j < i + \beta$, $(N^{-1}MN)(i,j) = 0$. After performing all the necessary computations we will obtain that for every $i, j \in A, i < j < i + \beta :$

$$(N^{-1}MN)(i,j) = \sum_{i \leq k \leq j} N(i,k)N^{-1}(k,j),$$

which is zero, being exactly $(NN^{-1})(i,j)$.

If β is a limit ordinal we prove the conclusion by transfinite induction after β. If $\beta = \omega$ than for every natural number $n \in \mathbb{N}$, for every $N \in \mathrm{UT}_A(R)$ and for every $M \in \mathrm{UT}_A^\beta(R)$ we have $N^{-1}MN \in \mathrm{UT}_A^n(R)$ (because we have already proven that the subgroups $\mathrm{UT}_A^n(R)$ are normal), so

$$N^{-1}MN \in \bigcap_{n \in \mathbb{N}} \mathrm{UT}_A^n(R) = \mathrm{UT}_A^\omega(R).$$

Now we suppose that for every ordinal number $\gamma < \beta$, for every $M \in \mathrm{UT}_A^\beta(R)$ and for every $N \in \mathrm{UT}_A(R)$ we have $N^{-1}MN \in \mathrm{UT}_A^\gamma(R)$. But we know that this property also holds for non-limit ordinals γ, therefore we deduce that for every $M \in \mathrm{UT}_A^\beta(R)$ and for every $N \in \mathrm{UT}_A(R)$

$$N^{-1}MN \in \bigcap_{\gamma < \beta} \mathrm{UT}_A^\gamma(R),$$

so $N^{-1}MN \in \mathrm{UT}_A^\beta(R)$. □

Corollary 3.4. *Consider the group $\mathrm{UT}_B(R)$, where the ordinal of A is the predecessor of B. Then α is the smallest ordinal for which $\mathrm{UT}_B^\alpha(R) = \mathrm{I}_{B,R}$, so the nilpotency class of $\mathrm{UT}_B(R)$ is exactly α.*

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