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GENERALIZATIONS OF HADAMARD PRODUCTS OF FUNCTIONS WITH NEGATIVE COEFFICIENTS. II

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Abstract. Let $T(n)$ be the class of functions with negative coefficients which are analytic in the unit disc U. For functions $f_1(z)$ and $f_2(z)$ belonging to $T(n)$, generalizations of the Hadamard product of $f_1(z)$ and $f_2(z)$ denoted by $f_1\Delta f_2(p,q;z)$ are introduced. In the present paper, some interesting properties of these generalizations of Hadamard products of functions in $T_n(\lambda,\alpha)$ and $C_n(\lambda,\alpha)$ are given.

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1. INTRODUCTION

Let $T(n)$ denote the class of functions of the form

$$f(z)=z-\sum_{k=n}^{\infty}a_kz^k \ (a_k\geq 0, n\in\mathbb{N}\setminus\{1\}=\{2,3,\dots\}) \tag{1.1}$$

which are analytic in the unit disc $U=\{z:|z|<1\}$.

Then a function $f(z)$ in $T(n)$ is said to be in the class $T_n(\lambda,\alpha)$ if satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\right\}>\alpha \tag{1.2}$$

for some α $(0\leq\alpha<1)$, λ $(0\leq\lambda<1)$ and for all $z\in U$.

Also, let $C_n(\lambda,\alpha)$ denote the subclass of $T(n)$ consisting of all functions satisfying the following condition

$$\operatorname{Re}\left\{\frac{f'(z)+f''(z)}{f'(z)+\lambda zf''(z)}\right\}>\alpha \tag{1.3}$$

for some α $(0\leq\alpha<1)$, λ $(0\leq\lambda<1)$ and for all $z\in U$.

Note that $T_2(0,\alpha)\equiv T^*(\alpha)$ and $C_2(0,\alpha)=C(\alpha)$, and that $f(z)\in C_n(\lambda,\alpha)$ if and only if $zf'(z)\in T_n(\lambda,\alpha)$. $T_n(0,\alpha)=T_n(\alpha)$ and $C_n(0,\alpha)=C_n(\alpha)$ studied by Duren [2] and Srivastava and Owa [3] .

Let $f_j(z)$ $(j=1,2)$ in $T(n)$ be given by

$$f_j(z)=z-\sum_{k=n}^{\infty}a_{k,j}\,z^k \ \ (n\geq 2, j=1,2). \tag{1.4}$$

Then the Hadamard product (or convolution) $f_1 * f_2$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=n}^{\infty} a_{k,1}\, a_{k,2}\, z^k. \tag{1.5}$$

For any real numbers p and q, we define the generalized Hadamard product $(f_1 \Delta f_2)$ by

$$(f_1 \Delta f_2)(p, q; z) = z - \sum_{k=n}^{\infty} (a_{k,1})^p (a_{k,2})^q z^k. \tag{1.6}$$

In the special case, if we take $p = q = 1$, then

$$(f_1 \Delta f_2)(1, 1; z) = (f_1 * f_2)(z) \quad (z \in U). \tag{1.7}$$

In the present paper, we make use of the generalized Hadamard product with a view to proving interesting characterization theorems involving the classes $T_n(\lambda, \alpha)$ and $C_n(\lambda, \alpha)$.

Note: Putting $\lambda = 0$ in all results we get:
$T_n(0, \alpha) = \tau * (n, \alpha)$ (Choi and Kim)[2]
$C_n(0, \alpha) = C\,(n, \alpha)$ (Choi and Kim)[2]

2. MAIN RESULTS

In order to prove our results for functions in the general classes $T_n(\lambda, \alpha)$ and $C_n(\lambda, \alpha)$, we shall need the following lemmas given by O. Altintas and S. Owa [1]:

Lemma 1. *A function $f(z)$ defined by* (1.1) *is in the class $T_n(\lambda, \alpha)$ if and only if*

$$\sum_{k=n}^{\infty} [k - \alpha(\lambda k + 1 - \lambda)] a_k \le 1 - \alpha. \tag{2.1}$$

Lemma 2. *A function $f(z)$ defined by* (1.1) *is in the class $C_n(\lambda, \alpha)$ if and only if*

$$\sum_{k=n}^{\infty} k[k - \alpha(\lambda k + 1 - \lambda)] a_k \le 1 - \alpha. \tag{2.2}$$

Remark 1. As pointed out earlier by O. Altintas and S. Owa [1], Lemma 1 and Lemma 2 follow immediately from a result due to O. Altintas and Owa [1] upon setting $a_k = 0$ $(k = 2, 3, \ldots, n - 1)$.

Applying Lemma 1 and Lemma 2, we shall prove the next result.

Theorem 1. *Let the functions* $f_j(z)$ $(j = 1,2)$ *defined by* (1.1) *be in the classes* $T_n(\lambda, \alpha_j)$ *for each* j, *then*

$$(f_1 \Delta f_2)\left(\frac{1}{p}, \frac{p-1}{p}; z\right) \in T_n(\lambda, \beta_1), \tag{2.3}$$

where $p > 1$ *and*

$$\beta_1 = \min_{k \geq n} \left\{ 1 - \frac{(k-1)(1-\lambda)}{\left(\frac{k-\alpha_1(\lambda k+1-\lambda)}{1-\alpha_1}\right)^{\frac{1}{p}} \left(\frac{k-\alpha_2(k\lambda+1-\lambda)}{1-\alpha_2}\right)^{1-\frac{1}{p}} - 1 - \lambda(k-1)} \right\}.$$

Proof. Since $f_j(z) \in T_n(\lambda, \alpha_j)$, by using Lemma 1 we have

$$\sum_{k=n}^{\infty} \frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j}\, a_{k,j} \leq 1 \quad (j = 1,2) \tag{2.4}$$

for $n \geq 2$. Moreover,

$$\left(\sum_{k=n}^{\infty} \frac{k-\alpha_1(\lambda k+1-\lambda)}{1-\alpha_1}\, a_{k,1}\right)^{\frac{1}{p}} \leq 1 \tag{2.5}$$

and

$$\left(\sum_{k=n}^{\infty} \frac{k-\alpha_2(\lambda k+1-\lambda)}{1-\alpha_2}\, a_{k,2}\right)^{\frac{p-1}{p}} \leq 1. \tag{2.6}$$

By the Hölder inequality, we get

$$\sum_{k=n}^{\infty} \left(\frac{k-\alpha_1(\lambda k+1-\lambda)}{1-\alpha_1}\right)^{\frac{1}{p}} \left(\frac{k-\alpha_2(\lambda k+1-\lambda)}{1-\alpha_2}\right)^{\frac{p-1}{p}} \cdot \\ \cdot (a_{k,1})^{1/p}(a_{k,2})^{p-1/p} \leq 1. \tag{2.7}$$

Since

$$(f_1 \Delta f_2)(\frac{1}{p}, \frac{p-1}{p}; z) = z - \sum_{k=n}^{\infty} (a_{k,1})^{1/p}(a_{k,2})^{p-1/p} z^k \quad (n \geq 2), \tag{2.8}$$

we see that

$$\sum_{k=n}^{\infty} \frac{k-\beta(\lambda k+1-\lambda)}{1-\beta}\, (a_{k,1})^{1/p}(a_{k,2})^{p-1/p} \leq 1 \quad (n \geq 2) \tag{2.9}$$

with

$$\beta \leq \min_{k\geq n}\left\{1-\frac{(k-1)(1-\lambda)}{\left(\frac{k-\alpha_1(\lambda k+1-\lambda)}{1-\alpha_1}\right)^{\frac{1}{p}}\left(\frac{k-\alpha_2(k\lambda+1-\lambda)}{1-\alpha_2}\right)^{1-\frac{1}{p}}-1-\lambda(k-1)}\right\}.$$

Thus, by Lemma 1, the proof of Theorem 1 is completed. □

COROLLARY 1. *If the functions $f_j(z)$ $(j=1,2)$ defined by* (1.1) *are in the class $T_n(\lambda,\alpha)$, then*

$$(f_1\Delta f_2)\left(\frac{1}{p},\frac{p-1}{p};z\right)\in T_n(\lambda,\alpha)\qquad (p>1). \tag{2.10}$$

Proof. In view of Lemma 1, corollary 1, follows readily from Theorem 1 in the special case $\alpha_j=\alpha$. □

THEOREM 2. *If the functions $f_j(z)$ $(j=1,2)$ defined by* (1.1) *are in the classes $C_n(\lambda,\alpha_j)$ for each j, then*

$$(f_1\Delta f_2)\left(\frac{1}{p},\frac{p-1}{p};z\right)\in C_n(\lambda,\beta), \tag{2.11}$$

where $p>1$ and

$$\beta = \min_{k\geq n}\left\{1-\frac{(k-1)(1-\lambda)}{\left(\frac{k-\alpha_1(\lambda k+1-\lambda)}{1-\alpha_1}\right)^{1/p}\left(\frac{k-\alpha_2(\lambda k+1-\lambda)}{1-\alpha_2}\right)^{1-\frac{1}{p}}-1-\lambda(k-1)}\right\}.$$

Proof. Since $f_j(z)\in C_n(\lambda,\alpha_j)$, by using Lemma 2, we get

$$\sum_{k=n}^{\infty}k\,\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j}\,a_{k.j}\leq 1\ \ (j=1,2),\ \ (n\geq 2). \tag{2.12}$$

Thus the proof of Theorem 2 in much again to that of Theorem 1 detailed already; instead of Lemma 1, it uses Lemma 2. □

COROLLARY 2. *If the functions $f_j(z)$ $(j=1,2)$ defined by* (1.1) *are in the class $C_n(\lambda,\alpha)$, then*

$$(f_1\Delta f_2)(\frac{1}{p},\frac{p-1}{p};z)\in C_n(\lambda,\alpha)\qquad (p>1). \tag{2.13}$$

THEOREM 3. *Let the functions $f_j(z)$ $(j=1,2,\ldots,m)$ defined by* (1.1) *be in the classes $T_n(\lambda,\alpha_j)$ for each j, and let $F_m(z)$ be defined by*

$$F_m(z)=z-\sum_{k=n}^{\infty}\left(\sum_{j=1}^{m}(a_{k,j})^p\right)z^k\ \ (n\geq p\geq 2,\ z\in U). \tag{2.14}$$

Then

$$F_m(z) \in T_n(\lambda, \beta_m) \qquad (n \geq 2), \tag{2.15}$$

where

$$\beta_m = 1 - (k-1)(1-\lambda)/(\frac{1}{m}(\frac{[k-\alpha(\lambda k+1-\lambda)]}{1-\alpha})^p - (\lambda k + 1 - \lambda))$$

and

$$n^{p-1}\left(\frac{n-\alpha(\lambda n+1-\lambda)}{(1-\alpha)}\right)^p \geq n\, m.$$

Proof. Since $f_j(z) \in T_n(\lambda, \alpha_j)$, using Lemma 1, we observe that

$$\sum_{k=n}^{\infty} \frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j}\, a_{k.j} \leq 1 \;\; (j = 1, 2, \dots, m,\; n \geq 2) \tag{2.16}$$

and

$$\begin{aligned}\sum_{k=n}^{\infty}\left(\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j}\right)^p (a_{k,j})^p \\ \leq \left(\sum_{k=n}^{\infty}\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j} a_{k,j}\right)^p \leq 1.\end{aligned} \tag{2.17}$$

If follows from (2.17) that

$$\sum_{k=n}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m}\left(\frac{k-\alpha_j(\lambda k+1-\lambda)}{(1-\alpha_j)}\right)^p (a_{k,j})^p\right\} \leq 1. \tag{2.18}$$

Putting

$$\alpha = \min_{1\leq j\leq m} \alpha_j, \tag{2.19}$$

and by virtue of Lemma 1, we find that

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{k-\beta_m(\lambda k+1-\lambda)}{1-\beta_m}\sum_{j=1}^{m}(a_{k,j})^p \\ \leq \sum_{k=n}^{\infty}\frac{1}{m}\left(\frac{k-\alpha(\lambda k+1-\lambda)}{1-\alpha}\right)^p\sum_{j=1}^{m}(a_{k,j})^p \\ \leq \sum_{k=n}^{\infty}\frac{1}{m}\sum_{j=1}^{m}\left(\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j}\right)^p (a_{k,j})^p \leq 1\end{aligned}$$

if

$$\beta_m \leq 1 - \frac{(k-1)(1-\lambda)}{\frac{1}{m}\left(\frac{k-\alpha(\lambda k+1-\lambda)}{1-\alpha}\right)^p - [\lambda k+1-\lambda]} \quad (k \geq n).$$

Now let

$$g(k) = 1 - \frac{(k-1)(1-\lambda)}{\frac{1}{m}(\frac{k-\alpha(\lambda k+1-\lambda)}{(1-\alpha)})^p - [\lambda k+1-\lambda]}. \tag{2.20}$$

Then $g'(k) \geq 0$ if $p \geq 2$. Hence

$$\beta_m \leq 1-(n-1)(1-\lambda) \,/\, \left(\frac{1}{m}\left(\frac{n-\alpha(\lambda n+1-\lambda)}{1-\alpha}\right)^p - (\lambda n+1-\lambda)\right). \tag{2.21}$$

By

$$n^{p-1}\left[\frac{n-\alpha(\lambda n+1-\lambda)}{(1-\alpha)}\right]^p \geq nm,$$

we see that $0 \leq \beta < 1$. Thus the proof of Theorem 3 is completed. □

THEOREM 4. *Let the functions* $f_j(z)$ $(j = 1, 2, \ldots, m)$ *defined by* (1.1) *be in the class* $C_n(\lambda, \alpha_j)$ *for each* j, *and let* $F_m(z)$ *be defined by* (2.14). *Then*

$$F_m(z) \in C_n(\lambda, \beta_m) \qquad (z\epsilon U), \tag{2.23}$$

where

$$\beta_m = 1 - (n-1)(1-\lambda) \;/\; \left(\frac{1}{m}\, n^{p-1}\left(\frac{n-\alpha(\lambda n+1-\lambda)}{1-\alpha}\right)^p - (\lambda n+1-\lambda)\right),$$

$$\alpha = \min_{1\leq j\leq m} a_j$$

and

$$n^{p-2}\left(\frac{n-\alpha(\lambda n+1-\lambda)}{(1-\alpha)}\right)^p \geq m.$$

Proof. Since $f_j(z) \in C_n(\lambda, \alpha_j)$, by using Lemma 2, we obtain

$$\sum_{k=n}^{\infty}(\frac{k[k-\alpha_j(k\lambda+1-\lambda)]}{(1-\alpha_j)}\, a_{k,j} \leq 1. \tag{2.23}$$

Thus the proof of Theorem 4 uses Lemma 2 in precisely the same manner as the above proof of Theorem 3 uses Lemma 1. The details may be omitted. □

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