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ON SOME CLASSES OF SETS VIA θ-GENERALIZED OPEN SETS

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Abstract. In this paper, we introduce and study the notions of θ-g-derived, θ-g-border, θ-g-frontier and θ-g-exterior of a set via the notion of θ-g-open sets. Nakaoka and Oda ([9] and [10]) introduced the notion of maximal open sets and minimal closed sets. By the same token, we introduce new classes of sets called maximal θ-g-open sets, minimal θ-g-closed sets, θ-g-semi maximal open sets and θ-g-semi minimal closed sets and investigate some of their fundamental properties.

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1. INTRODUCTION

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc by utilizing generalized open sets. One of the most well-known notions and also an inspiration source is the notion of θ-open sets introduced by N. V. Veličko [12] in 1968. In 1943, Fomin [6] (see, also [7]) introduced the notion of θ-continuity. The notions of θ-closed subsets and the θ-closure were also introduced by Veličko [12] for the purpose of studying the important class of H-closed spaces in terms of arbitrary filterbases. Dickman and Porter [3], [4] and Joseph [8] continued the work of Veličko. Recently Noiri and Jafari [11] have also obtained several new and interesting results related to these sets. Quite recently, Caldas et al. [[1], [2]] introduced and studied the notions of Λ_θ-sets, (Λ, θ)-closed sets and (Λ, θ)-open sets by utilizing θ-open sets and θ-closed sets.

In what follows (X, τ) and (Y, σ) (or X and Y) denote topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$, respectively. A point $x \in X$ is called a θ-cluster point of A if $A \cap Cl(U) \neq \emptyset$ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $Cl_\theta(A)$. A subset A is called θ-closed if $A = Cl_\theta(A)$. The complement of a θ-closed set is called θ-open. We denote the collection of all θ-open (respectively, θ-closed) sets by $\theta(X, \tau)$ (respectively, $cl\theta(X, \tau)$). A subset A of a topological space (X, τ) is called θ-generalized closed ($= \theta$-g-closed) [5] if $Cl_\theta(A) \subseteq U$, whenever $A \subseteq U$ and U is open in (X, τ). Hence the union of two θ-g-closed sets is a θ-g-closed set and the intersection of two θ-g-closed sets is generally not a θ-g-closed set. By ([5], Theorem 3.12(iii)), the finite intersection of θ-g-closed sets is not always θ-g-closed. In a T_1-space θ-g-closed are θ-closed and in a $T_{\frac{1}{2}}$-space any θ-g-closed set is a closed set. The complement of a θ-g-closed set is called θ-g-open equivalently A is θ-g-open if $F \subset Int_\theta(A)$ whenever F is closed and $F \subset A$. If A is θ-g-open in X and B is θ-g-open in Y, then $A \times B$ is θ-g-open in $X \times Y$ [5]. The union of any θ-g-open sets is not always θ-g-open.

A proper nonempty open set (resp. closed set) U of X (resp. V of X) is said to be a maximal open set [9] (resp. minimal closed set [10]) if any open (resp. closed) set which contains U is either X or U (resp.contained in V is either $\emptyset$ or V). The purpose of the present paper is to offer and study some new notions such as θ-g-derived, θ-g-border, θ-g-frontier and θ-g-exterior of a set via the notion of θ-g-open sets. We also introduce and investigate new classes of sets called maximal θ-g-open sets, minimal θ-g-closed sets, θ-g-semi maximal open sets and θ-g-semi minimal closed sets vis θ-g-open sets and θ-g-closed sets.

2. PROPERTIES OF θ-G-OPEN SETS

Definition 1. The intersection of all θ-g-closed sets containing a set A is called the θ-g-closure of A and is denoted by $\theta Cl_g(A)$. This is, for any $A \subset X$, $\theta Cl_g(A) = \bigcap\{F \in \Gamma : A \subset F\}$ where $\Gamma = \{F : F \subset X \textit{ and } F \textit{ is } \theta\textit{-}g\textit{-closed}\}$.

The collection of all θ-g-closed (resp. θ-g-open) subsets of X will be denoted by $\theta GC(X)$ (resp. $\theta GO(X)$). We set $\theta GC(X, x) = \{V \in \theta GC(X) : x \in V\}$ for $x \in X$. We define similarly $\theta GO(X, x)$.

Theorem 2.1. *For any subset A of a space X, the following statements hold:*
(1) $A \subset \theta Cl_g(A) \subset Cl_\theta(A)$.
(2) $\theta Cl_g(A)$ *is not always θ-g-closed.*
(3) $x \in \theta Cl_g(A)$ *if and only if for any θ-g-open set U containing x, $A \bigcap U \neq \emptyset$.*

Proof. (1) It suffices to observe that every θ-closed is θ-g-closed.
(3) Necessity. Suppose that $x \in \theta Cl_g(A)$. Let U be a θ-g-open set containing x such that $A \bigcap U = \emptyset$. And so, $A \subset X\backslash U$. But $X\backslash U$ is θ-g-closed and hence $\theta Cl_g(A) \subset X\backslash U$. Since $x \notin X\backslash U$, we obtain $x \notin \theta Cl_g(A)$ which is contrary to the hypothesis.

Sufficiency. Suppose that every θ-g-open set of X containing x meets A. If $x \notin \theta Cl_g(A)$, then there exists a θ-g-closed set F of X such that $A \subset F$ and $x \notin F$. Therefore $x \in X\backslash F \in \theta GO(X)$. Hence $X\backslash F$ is a θ-g-open set of X containing x, but $(X\backslash F) \bigcap A = \emptyset$. This is contrary to the hypothesis. □

In general the converse of Theorem 2.1(1) may not be true.

Example 2.2. Let $X = \{a, b, c, d\}$ with the topology

$$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}.$$

Then $\{\emptyset, X\}$ is the set of all θ-closed sets in (X, τ) and
$\theta GC(X, \tau) = \{\emptyset, \{b, c\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}, \{a, b, c\}, X\}$. Hence:
(i) $Cl_\theta(\{b\}) = X$, $\theta Cl_g(\{b\}) = \{a, c\}$ and $Cl_\theta(\{b\}) \not\subseteq \theta Cl_g(\{b\})$.
(ii) $\theta Cl_g(\{c, d\}) = \{b, c, d\}$, $Cl(\{c, d\}) = \{c, d\}$ and $\theta Cl_g(\{c, d\}) \not\subseteq Cl(\{c, d\})$.
$\theta Cl_g(\{b, d\}) = \{b, d\}$, $Cl(\{b, d\}) = \{b, c, d\}$ and $Cl(\{b, d\}) \not\subseteq \theta Cl_g(\{b, d\})$.

EXAMPLE 2.3. Let (X, τ) be the space in the example above. Then the set $\theta Cl_g(\{b\}) = \{b\}$ is not θ-g-closed.

DEFINITION 2. Let A be a subset of a space X. A point $x \in A$ is said to be a θ-g-limit point of A if for each θ-g-open set U containing x , $U \cap (A \backslash \{x\}) \neq \emptyset$. The set of all θ-g-limit points of A is called the θ-g-derived set of A and is denoted by $\theta D_g(A)$.

THEOREM 2.4. *For subsets A, B of a space X, the following statements hold:*
(1) *$\theta D_g(A) \subset D_\theta(A)$, where $D_\theta(A)$ is the θ-derived set of A.*
(2) *If $A \subset B$, then $\theta D_g(A) \subset \theta D_g(B)$.*
(3) *$\theta D_g(A) \cup \theta D_g(B) \subset \theta D_g(A \cup B)$ and $\theta D_g(A \cap B) \subset \theta D_g(A) \cap \theta D_g(B)$.*
(4) *$\theta D_g(\theta D_g(A)) \backslash A \subset \theta D_g(A)$.*
(5) *$\theta D_g(A \cup \theta D_g(A)) \subset A \cup \theta D_g(A)$.*

Proof. (1) It suffices to observe that every θ-open set is θ-g-open.
(3) Follows by (2).
(4) If $x \in \theta D_g(\theta D_g(A)) \backslash A$ and U is an θ-g-open set containing x, then $U \cap (\theta D_g(A) \backslash \{x\}) \neq \emptyset$. Let $y \in U \cap (\theta D_g(A) \backslash \{x\})$. Then since $y \in \theta D_g(A)$ and $y \in U$, $U \cap (A \backslash \{y\}) \neq \emptyset$. Let $z \in U \cap (A \backslash \{y\})$. Then $z \neq x$ for $z \in A$ and $x \notin A$. Hence $U \cap (A \backslash \{x\}) \neq \emptyset$. Therefore $x \in \theta D_g(A)$.
(5) Let $x \in \theta D_g(A \cup \theta D_g(A))$. If $x \in A$, the result is obvious. So let $x \in \theta D_g(A \cup \theta D_g(A)) \backslash A$, then for any θ-g-open set U containing x, $U \cap (A \cup \theta D_g(A) \backslash \{x\}) \neq \emptyset$. Thus $U \cap (A \backslash \{x\}) \neq \emptyset$ or $U \cap (\theta D_g(A) \backslash \{x\}) \neq \emptyset$. Now it follows similarly from (4) that $U \cap (A \backslash \{x\}) \neq \emptyset$. Hence $x \in \theta D_g(A)$. Therefore, in any case $\theta D_g(A \cup \theta D_g(A)) \subset A \cup \theta D_g(A)$. □

In general the converse of Theorem 2.4(1) may not be true and the equality does not hold in (3) of Theorem 2.4.

EXAMPLE 2.5. A counterexample illustrating that $\theta D_g(A \cap B) \neq \theta D_g(A) \cap \theta D_g(B)$ in general can be easily found in regular T_1-spaces (e.g. in $\mathbf{R}$), for which open, θ-open and θ-g-open sets (and hence D, D_θ and θD_g) coincide.

THEOREM 2.6. *For any subset A of a space X, $\theta Cl_g(A) = A \cup \theta D_g(A)$.*

Proof. Since $\theta D_g(A) \subset \theta Cl_g(A)$, $A \cup \theta D_g(A) \subset \theta Cl_g(A)$. On the other hand, let $x \in \theta Cl_g(A)$. If $x \in A$, then the proof is complete. If $x \notin A$, each θ-g-open set U containing x intersects A at a point distinct from x, so $x \in \theta D_g(A)$. Thus $\theta Cl_g(A) \subset A \cup \theta D_g(A)$, which completes the proof. □

DEFINITION 3. An point $x \in X$ is said to be a θ-g-interior point of A if there exists an θ-g-open set U containing x such that $U \subset A$. The set of all θ-g-interior points of A is called the θ-g-interior of A and denoted by $\theta Int_g(A)$.

THEOREM 2.7. *For subsets A, B of a space X, the following statements are true:*
(1) $\theta Int_g(A) = \bigcup\{U \mid U \subset A, U \in \theta GO(X)\}$.
(2) *If A is θ-g-open then* $A = \theta Int_g(A)$.
(3) *If A is θ-g-open then* $\theta Int_g(\theta Int_g(A)) = \theta Int_g(A)$.
(4) $A \backslash \theta D_g(X \backslash A) \subset \theta Int_g(A)$.
(5) $X \backslash \theta Int_g(A) = \theta Cl_g(X \backslash A)$.
(6) $X \backslash \theta Cl_g(A) = \theta Int_g(X \backslash A)$.
(7) $A \subset B$, *then* $\theta Int_g(A) \subset \theta Int_g(B)$.
(8) $\theta Int_g(A) \cup \theta Int_g(B) \subset \theta Int_g(A \cup B)$.
(9) $\theta Int_g(A) \cap \theta Int_g(B) \supset \theta Int_g(A \cap B)$.

Proof. (4) If $x \in A \backslash \theta D_g(X \backslash A)$, then $x \notin \theta D_g(X \backslash A)$ and so there exists a θ-g-open set U containing x such that $U \cap (X \backslash A) = \emptyset$. Then $x \in U \subset A$ and hence $x \in \theta Int_g(A)$, i.e., $A \backslash \theta D_g(X \backslash A) \subset \theta Int_g(A)$.
(5) $X \backslash \theta Int_g(A) = \cap\{F \in X \mid A \subset F, (F = \theta\text{-}g\text{-}closed)\} = \theta Cl_g(X \backslash A)$. □

DEFINITION 4. $\theta b_g(A) = A \backslash \theta Int_g(A)$ is called the θ-g-border of A.

THEOREM 2.8. *For a subset A of a space X, the following statements hold:*
(1) $\theta b_g(A) \subset b_\theta(A)$, *where $b_\theta(A)$ denotes the θ-border of A.*
(2) $A = \theta Int_g(A) \cup \theta b_g(A)$.
(3) $\theta Int_g(A) \cap \theta b_g(A) = \emptyset$.
(4) *If A is θ-g-open, then* $\theta b_g(\theta Int_g(A)) = \emptyset$.
(5) $\theta Int_g(\theta b_g(A)) = \emptyset$.
(6) $\theta b_g(\theta b_g(A)) = \theta b_g(A)$.
(7) $\theta b_g(A) = A \cap \theta Cl_g(X \backslash A)$.

Proof. (5) If $x \in \theta Int_g(\theta b_g(A))$, then $x \in \theta b_g(A)$. On the other hand, since $\theta b_g(A) \subset A$, $x \in \theta Int_g(\theta b_g(A)) \subset \theta Int_g(A)$. Hence $x \in \theta Int_g(A) \cap \theta b_g(A)$ which contradicts (3). Thus $\theta Int_g(\theta b_g(A)) = \emptyset$.
(7) $\theta b_g(A) = A \backslash \theta Int_g(A) = A \backslash (X \backslash \theta Cl_g(X \backslash A)) = A \cap \theta Cl_g(X \backslash A)$. □

EXAMPLE 2.9. Consider the topological space (X, τ) given in Example 2.2, where $\theta GC(X, \tau) = \{\emptyset, \{b, c\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}, \{a, b, c\}, X\}$. If $A = \{b, c\}$. Then $\theta b_g(A) = \{b\}$ and $b_\theta(A) = \{b, c\}$. Hence $b_\theta(A) \not\subseteq \theta b_g(A)$, i.e., in general the opposite implication of Theorem 2.8 (1) may not be true.

DEFINITION 5. $\theta Fr_g(A) = \theta Cl_g(A) \backslash \theta Int_g(A)$ is called the θ-g-frontier of A.

THEOREM 2.10. *For a subset A of a space X, the following statements hold:*
(1) $\theta Fr_g(A) \subset Fr_\theta(A)$, *where $Fr_\theta(A)$ denotes the θ-frontier of A.*
(2) $\theta Cl_g(A) = \theta Int_g(A) \cup \theta Fr_g(A)$.

(3) $\theta Int_g(A) \cap \theta Fr_g(A) = \emptyset$.
(4) $\theta b_g(A) \subset \theta Fr_g(A)$.
(5) *If* A *is a* θ-g-*open set, then* $\theta Fr_g(A) = \theta D_g(A)$.
(6) $\theta Fr_g(A) = \theta Cl_g(A) \cap \theta Cl_g(X \backslash A)$.
(7) $\theta Fr_g(A) = \theta Fr_g(X \backslash A)$.
(8) $\theta Fr_g(\theta Int_g(A)) \subset \theta Fr_g(A)$.
(9) $\theta Fr_g(\theta Cl_g(A)) \subset \theta Fr_g(A)$.
(10) $\theta Int_g(A) = A \backslash \theta Fr_g(A)$.

Proof. (2) $\theta Int_g(A) \cup \theta Fr_g(A) = \theta Int_g(A) \cup (\theta Cl_g(A) \backslash \theta Int_g(A)) = \theta Cl_g(A)$.
(3) $\theta Int_g(A) \cap \theta Fr_g(A) = \theta Int_g(A) \cap (\theta Cl_g(A) \backslash \theta Int_g(A)) = \emptyset$.
(6) $\theta Fr_g(A) = \theta Cl_g(A) \backslash \theta Int_g(A) = \theta Cl_g(A) \cap \theta Cl_g(X \backslash A)$.
(9) We have that

$$\begin{aligned} &\theta Fr_g(\theta Cl_g(A)) = \theta Cl_g(\theta Cl_g(A)) \backslash \theta Int_g(\theta Cl_g(A)) \\ &= \theta Cl_g((A)) \backslash \theta Int_g(\theta Cl_g(A)) \subset \theta Cl_g(A) \backslash \theta Int_g(A) = \theta Fr_g(A). \end{aligned}$$

(10) $A \backslash \theta Fr_g(A) = A \backslash (\theta Cl_g(A) \backslash \theta Int_g(A)) = \theta Int_g(A)$. □

The converses of (1) and (4) of Theorem 2.10 are not true in general, as shown by Example 2.11.

EXAMPLE 2.11. Consider the topological space (X, τ) given in Example 2.2. If $A = \{d\}$, then $Fr_\theta(A) = X$, $\theta Fr_g(A) = \{b\}$, $\theta b_g(A) = \emptyset$. Therefore $Fr_\theta(A) \not\subseteq \theta Fr_g(A)$ and $\theta Fr_g(A) \not\subseteq \theta b_g(A)$.

Recall, that a mapping $f : X \to Y$ from a topological space X into a topological space Y is called θ-g-continuous, [5] if the inverse image of every closed set in Y is θ-g-closed in X.

THEOREM 2.12. *Assume that* $\theta GO(X)$ *is closed by unions. Then the following are equivalent for a function* $f : X \to Y$*:*
(1) f *is* θ-g-*continuous;*
(2) *for every open subset* V *of* Y, $f^{-1}(V) \in \theta GO(X)$*;*
(3) *for each* $x \in X$ *and each* $V \in O(Y, f(x))$, *there exists* $U \in \theta GO(X, x)$ *such that* $f(U) \subset V$.

Proof. (1) $\leftrightarrow$ (2) : This follows for $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$.
(2) $\to$ (3) : Let $V \in O(Y)$ and $f(x) \in V$. Since f is θ-g-continuous, $f^{-1}(V) \in \theta GO(X)$ and $x \in f^{-1}(V)$. Put $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subset V$.
(3) $\to$ (2) : Let V be an open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$. Therefore by (3) there exists a $U_x \in \theta GO(X)$ such that $x \in U_x$ and $f(U_x) \subset V$. Therefore $x \in U_x \subset f^{-1}(V)$. This implies that $f^{-1}(V)$ is a union of θ-g-open sets of X. Consequently $f^{-1}(V) \in \theta GO(X)$. Hence f is θ-g-continuous. □

In the following theorem N_θ-$g.c.$ denotes the set of points x of X for which a function $f : (X, \tau) \to (Y, \sigma)$ is not θ-g-continuous.

THEOREM 2.13. *Assume that $\theta GO(X)$ is closed by unions. Then N_θ-g.c. is identical with the union of the θ-g-frontiers of the inverse images of θ-g-open sets containing $f(x)$.*

Proof. Suppose that f is not θ-g-continuous at a point x of X. Then there exists an open set $V \subset Y$ containing $f(x)$ such that $f(U)$ is not a subset of V for every $U \in \theta GO(X,x)$. Hence we have $U \cap (X \setminus f^{-1}(V)) \neq \emptyset$ for every $U \in \theta GO(X)$ containing x. It follows that $x \in \theta Cl_g(X \setminus f^{-1}(V))$. We also have $x \in f^{-1}(V) \subset \theta Cl_g(f^{-1}(V))$. This means that $x \in \theta Fr_g(f^{-1}(V))$.

Now, let f be θ-g-continuous at $x \in X$ and $V \subset Y$ be any open set containing $f(x)$. Then $x \in f^{-1}(V)$ is a θ-g-open set of X. Thus $x \in \theta Int_g(f^{-1}(V))$ and therefore $x \notin \theta Fr_g(f^{-1}(V))$ for every open set V containing $f(x)$. □

DEFINITION 6. $\theta Ext_g(A) = \theta Int_g(X \backslash A)$ is called the θ-g-exterior of A.

THEOREM 2.14. *For a subset A of a space X, the following statements hold:*
(1) $\theta Ext(A) \subset \theta Ext_g(A)$, *where $\theta Ext(A)$ denotes the θ-exterior of A.*
(2) $\theta Ext_g(A) = \theta Int_g(X \backslash A) = X \backslash \theta Cl_g(A)$.
(3) $\theta Ext_g(\theta Ext_g(A)) = \theta Int_g(\theta Cl_g(A))$.
(4) *If* $A \subset B$, *then* $\theta Ext_g(A) \supset \theta Ext_g(B)$.
(5) $\theta Ext_g(A \cup B) \subset \theta Ext_g(A) \cup \theta Ext_g(B)$.
(6) $\theta Ext_g(A \cap B) \supset \theta Ext_g(A) \cap \theta Ext_g(B)$.
(7) $\theta Ext_g(X) = \emptyset$.
(8) $\theta Ext_g(\emptyset) = X$.
(9) $\theta Int_g(A) \subset \theta Ext_g(\theta Ext_g(A))$.
(10) $X = \theta Int_g(A) \cup \theta Ext_g(A) \cup \theta Fr_g(A)$.

Proof. (3) Note that

$$\begin{aligned}&\theta Ext_g(\theta Ext_g(A)) = \theta Ext_g(X \backslash \theta Cl_g(A)) = \theta Int_g(X \backslash (X \backslash \theta Cl_g(A)))\\&= \theta Int_g(\theta Cl_g(A))\end{aligned}$$

(9) The following relations hold

$$\begin{aligned}&\theta Int_g(A) \subset \theta Int_g(\theta Cl_g(A)) = \theta Int_g(X \backslash \theta Int_g(X \backslash A))\\&= \theta Int_g(X \backslash \theta Ext_g(A)) = \theta Ext_g(\theta Ext_g(A)).\end{aligned}$$ □

3. NEW CLASSES OF SETS VIA θ-G-CLOSED AND θ-G-OPEN SETS

DEFINITION 7. A proper nonempty θ-g-open set A of X is said to be a maximal θ-g-open set if any θ-g-open set which contains A is either X or A.

DEFINITION 8. A proper nonempty θ-g-closed set B of X is said to be a minimal θ-g-closed set if any θ-g-closed set which is contained in B is either $\emptyset$ or B.

THEOREM 3.1. *A proper nonempty subset A of X is maximal θ-g-open if and only if $X \backslash A$ is a minimal θ-g-closed set.*

Proof. Necessity. Let A be a maximal θ-g-open set. Suppose that B is a θ-g-closed set such that $B \subset X\backslash A$. Then $A \subset X\backslash B$ and $X\backslash B$ is θ-g-open. Since A is maximal θ-g-open, we have $A = X\backslash B$ or $X = X\backslash B$ and hence $B = X\backslash A$ or $B = \emptyset$. This shows that $X\backslash A$ is minimal θ-g-closed.

Sufficiency. The proof is similar to that of Necessity.

Definition 9. A set A in a topological space X is said to be a θ-g-semi-maximal open if there exists a maximal θ-g-open set U such that $U \subset A \subset Cl(U)$. The complement of a θ-g-semi-maximal open set is called a θ-g-semi-minimal closed set.

Remark 3.2. Every maximal θ-g-open (resp. minimal θ-g-closed) set is θ-g-semi-maximal open (resp. θ-g-semi-minimal closed).

Theorem 3.3. *If A is a θ-g-semi-maximal open set of X and $A \subset B \subset Cl(A)$. Then B is a θ-g-semi-maximal open set of X.*

Proof. Since A is θ-g-semi-maximal open, there exists a maximal θ-g-open set U such that $U \subset A \subset Cl(U)$. Then $U \subset A \subset B \subset Cl(A) \subset Cl(U)$. Hence $U \subset B \subset Cl(U)$. Thus B is θ-g-semi-maximal open. □

Theorem 3.4. *A subset F of X is θ-g-semi-minimal closed if and only if there exists a minimal θ-g-closed set G in X such that $Int(G) \subset F \subset G$.*

Proof. Suppose F is θ-g-semi-minimal closed in X. By Definition 9, $X\backslash F$ is θ-g-semi-maximal open in X. Therefore, there exists a maximal θ-g-open set U such that $U \subset X\backslash F \subset Cl(U)$, which implies $Int(X\backslash U) = X\backslash Cl(U) \subset F \subset X\backslash U$. Take $G = X\backslash U$, so that G is a minimal θ-g-closed set, such that $Int(G) \subset F \subset G$.

Conversely, Suppose that there exists a minimal θ-g-closed set G in X, such that $Int(G) \subset F \subset G$. Hence $X\backslash G \subset X\backslash F \subset X\backslash Int(G) = Cl(X\backslash G)$. Therefore there exists a maximal θ-g-open set $U = X\backslash G$ such that $U \subset X\backslash F \subset Cl(U)$, i.e., $X\backslash F$ is θ-g-semi-maximal open in X. It follows that F is θ-g-semi-minimal closed. □

Theorem 3.5. *If G is θ-g-semi-minimal closed in X and if $Int(G) \subset F \subset G$, then F is also θ-g-semi-minimal closed in X.*

Proof. Let G be a θ-g-semi-minimal closed set of X . Then there exists a minimal θ-g-closed set H in X, such that $Int(H) \subset G \subset H$. Hence $Int(H) \subset Int(G) \subset F \subset G \subset H$. It follows $Int(H) \subset F \subset H$. Therefore F is a θ-g-semi-minimal closed set of X. □

We close with the following questions:

Question 3.6. Is it true that $\theta Fr_g(A) = \theta b_g(A) \cup \theta D_g(A)$?

Question 3.7. Let Y be an open subspace of X and $A \subset Y$. Is it true that if A is a θ-g-semi-maximal open set of Y, then A is a θ-g-semi-maximal open set of Y?

QUESTION 3.8. Is it true that if A_i is a θ-g-semi-maximal open set of X_i ($i = 1, 2$), then $A_1 \times A_2$ is a θ-g-semi-maximal open set of $X_1 \times X_2$?

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