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MAXWELL EQUATIONS ON THE SECOND ORDER TANGENT BUNDLE

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Abstract. We generalize the geometrical theory of electromagnetic fields in [7] to the second order tangent bundle T^2M endowed with an arbitrary N-linear connection and, by defining the current density J, we give an analoguous of the charge conservation law in the second order differential geometry.

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1. INTRODUCTION

Starting from the tensorial form of the first Maxwell equations (Gauss' law for magnetism and Faraday's law of induction), in [7], R. Miron and Gh. Atanasiu constructed an electromagnetic field theory on the k-tangent (or k-osculator) bundle endowed with a particular nonlinear connection N and a particular linear connection $C\Gamma(N)$. On the other hand, in [14] there is defined the current density and studied its divergence on the tangent bundle of order 1, TM, also endowed with a particular linear connection.

In the following, we first aim to generalize the construction in [7] in the case of an arbitrary nonlinear connection N on the second order tangent bundle and an arbitrary metrical linear connection which preserves the distributions generated by N. Then, we define a notion of current density on the second order tangent bundle T^2M which generalizes the one in [14], write the second Maxwell equations (the analoguous of Gauss' law for magnetism and of Ampere's law) and the charge conservation law in our geometrical context.

2. THE 2-TANGENT BUNDLE T^2M

Let M be a real n-dimensional manifold of class $\mathcal{C}^\infty$, $\left(T^2M, \pi^2, M\right)$ its second order jet bundle, called in the subsequent, as in [1], the second order tangent bundle, and let $\widetilde{T^2M}$ be the space T^2M without its null section. For a point $u \in T^2M$, let $(x^i, y^{(1)i}, y^{(2)i})$ be its coordinates in a local chart.

Let N be a nonlinear connection, [5], [8]–[13], and let $\left(\underset{1}{N}{}^i_j, \underset{2}{N}{}^i_j\right)$, i, $j = 1, \ldots, n$ be its coefficients. Then, N determines the direct decomposition

$$T_uT^2M = N_0(u) \oplus N_1(u) \oplus V_2(u), \; \forall u \in T^2M. \tag{1}$$

We denote the adapted basis to (1) by $(\delta_i, \delta_{1i}, \delta_{2i})$ and its dual basis with $(dx^i, \delta y^{(1)i}, \delta y^{(2)i})$. We have

$$(2)\quad \begin{cases} \delta_i = \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - \underset{1}{N}{}_i^k \dfrac{\partial}{\partial y^{(1)k}} - \underset{2}{N}{}_i^k \dfrac{\partial}{\partial y^{(2)k}} \\ \delta_{1i} = \dfrac{\delta}{\delta y^{(1)i}} = \dfrac{\partial}{\partial y^{(1)i}} - \underset{1}{N}{}_i^k \dfrac{\partial}{\partial y^{(2)k}} \\ \delta_{2i} = \dfrac{\partial}{\partial y^{(2)i}}, \end{cases}$$

respectively,

$$(3)\quad \begin{cases} \delta y^{(1)i} = dy^{(1)i} + \underset{1}{M}{}_k^i dx^k \\ \delta y^{(2)i} = dy^{(2)i} + \underset{1}{M}{}_k^i dy^{(1)k} + \underset{2}{M}{}_k^i dx^k, \end{cases}$$

where $\underset{1}{M}{}_k^i, \underset{2}{M}{}_k^i$ are the dual coefficients of the nonlinear connection N.

Then, a vector field $X \in \mathcal{X}\left(T^2M\right)$ is represented in the local adapted basis as

$$(4)\quad X = X^{(0)i}\delta_i + X^{(1)i}\delta_{1i} + X^{(2)i}\delta_{2i},$$

with the three right terms,

$$(5)\quad hX = X^H = X^{(0)i}\delta_i,\ v_1X = X^{V_1} = X^{(1)i}\delta_{1i},\ v_2X = X^{V_2} = X^{(2)i}\delta_{2i},$$

called ***d*-vector fields**, belonging to the distributions N, N_1 and V_2 respectively.

A 1-form $\omega \in \mathcal{X}^*\left(T^2M\right)$ will be decomposed as

$$\omega = \omega_i^{(0)} dx^i + \omega_i^{(1)} \delta y^{(1)i} + \omega_i^{(2)} \delta y^{(2)i}.$$

The terms

$$\omega^H = \omega_i^{(0)} dx^i, \omega^{V_1} = \omega_i^{(1)} \delta y^{(1)i}, \omega^{V_2} = \omega_i^{(2)} \delta y^{(2)i}$$

are called ***d*-covector fields**.

A ***d*-tensor field** is a tensor field of type (r, s) on T^2M which acts on r d-covector fields and s d-vector fields, in the following manner:

$$T(\underset{1}{\omega}, ..., \underset{r}{\omega}, \overset{1}{X}, ..., \overset{s}{X}) = T(\underset{1}{\omega}{}^H, ..., \underset{r}{\omega}{}^{V_2}, \overset{1}{X}{}^H, ..., \overset{s}{X}{}^{V_2}).$$

Any tensor field $T \in \mathcal{T}_s^r\left(T^2M\right)$ can be split with respect to (1) into a sum of d-tensor fields.

The $\mathcal{F}\left(T^2M\right)$-linear mapping $J : \mathcal{X}\left(T^2M\right) \to \mathcal{X}\left(T^2M\right)$ given by

$$(6)\quad J(\delta_i) = \delta_{1i},\ J(\delta_{1i}) = \delta_{2i},\ J(\delta_{2i}) = 0,$$

is called the **2-tangent structure** on T^2M, [8]–[13].

The **Liouville vector field**, [1], [5],

$$\overset{2}{\mathbb{C}} = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(2)i}},$$

can be written in the adapted basis (2) as

$$\overset{2}{\mathbb{C}} = z^{(1)i}\delta_{1i} + 2z^{(2)i}\delta_{2i}.$$

Its components

(7) $$z^{(1)i} = y^{(1)i}, z^{(2)i} = y^{(2)i} + \frac{1}{2}\underset{1}{M}{}^i_j y^{(1)j},$$

define two d-vector fields, called **the Liouville d-vector fields.**

3. N-LINEAR CONNECTIONS

An **N-linear connection** D, [1], is a linear connection on T^2M, which preserves by parallelism the distributions N, N_1 and V_2. An N-linear connection which is also compatible to J ($DJ = 0$) is called, [1], a **JN-linear connection.**

An N-linear connection is locally given by its nine coefficients

(8)
$$D\Gamma(N) = \left(\underset{(00)}{L}{}^i{}_{jk}, \underset{(10)}{L}{}^i{}_{jk}, \underset{(20)}{L}{}^i{}_{jk}, \underset{(01)}{C}{}^i{}_{jk}, \underset{(11)}{C}{}^i{}_{jk}, \underset{(21)}{C}{}^i{}_{jk}, \underset{(02)}{C}{}^i{}_{jk}, \underset{(12)}{C}{}^i{}_{jk}, \underset{(22)}{C}{}^i{}_{jk} \right),$$

where

(9) $$\begin{cases} D_{\delta_k}\delta_j = \underset{(00)}{L}{}^i{}_{jk}\delta_i,\ D_{\delta_k}\delta_{1j} = \underset{(10)}{L}{}^i{}_{jk}\delta_{1i},\ D_{\delta_k}\delta_{2j} = \underset{(20)}{L}{}^i{}_{jk}\delta_{2i} \\ D_{\delta_{1k}}\delta_j = \underset{(01)}{C}{}^i{}_{jk}\delta_i,\ D_{\delta_{1k}}\delta_{1j} = \underset{(11)}{C}{}^i{}_{jk}\delta_{1i},\ D_{\delta_{1k}}\delta_{2j} = \underset{(21)}{C}{}^i{}_{jk}\delta_{2i} \\ D_{\delta_{2k}}\delta_j = \underset{(02)}{C}{}^i{}_{jk}\delta_i,\ D_{\delta_{2k}}\delta_{1j} = \underset{(12)}{C}{}^i{}_{jk}\delta_{1i},\ D_{\delta_{2k}}\delta_{2j} = \underset{(22)}{C}{}^i{}_{jk}\delta_{2i} \end{cases}.$$

In the particular case when D is J-compatible, we have only three essential coefficients:

$$\begin{aligned} \underset{(00)}{L}{}^i{}_{jk} &= \underset{(10)}{L}{}^i{}_{jk} = \underset{(20)}{L}{}^i{}_{jk} =: L^i{}_{jk}, \\ \underset{(01)}{C}{}^i{}_{jk} &= \underset{(11)}{C}{}^i{}_{jk} = \underset{(21)}{C}{}^i{}_{jk} =: \underset{(1)}{C}{}^i{}_{jk}, \\ \underset{(02)}{C}{}^i{}_{jk} &= \underset{(12)}{C}{}^i{}_{jk} = \underset{(22)}{C}{}^i{}_{jk} =: \underset{(2)}{C}{}^i{}_{jk}. \end{aligned}$$

Let

$$T = T^{i_1\ldots i_r}_{j_1\ldots j_s}\left(x, y^{(1)}, y^{(2)}\right)\delta_{i_1} \otimes \ldots \otimes \delta_{2i_r} \otimes dx^{j_1} \otimes \ldots \otimes \delta y^{(2)j_s}$$

be a d-tensor field of type (r, s) and $X \in \mathcal{X}(T^2M)$, $X = X^H + X^{V_1} + X^{V_2}$ as in (4). Then, the covariant derivative of T writes as

$$D_XT = D^H_XT + D^{V_1}_XT + D^{V_2}_XT,$$

where the h-, v_1- and v_2- **covariant derivatives** D^H_XT, $D^{V_1}_XT$, $D^{V_2}_XT$ are given by:

$$(D_X^H T)(\underset{1}{\omega}^H, ..., \underset{r}{\omega}^{V_2}, \overset{1}{X}^H, ..., \overset{s}{X}^{V_2}) = X^H(T(\underset{1}{\omega}^H, ..., \underset{r}{\omega}^{V_2}, \overset{1}{X}^H, ..., \overset{s}{X}^{V_2})-$$

$$-T(D_X^H \underset{1}{\omega}^H, ..., \underset{r}{\omega}^{V_2}, \overset{1}{X}^H, ..., \overset{s}{X}^{V_2}) - ... - T(\underset{1}{\omega}^H, ..., \underset{r}{\omega}^{V_2}, \overset{1}{X}^H, ..., D_X^H \overset{s}{X}^{V_2}),$$

$$(D_X^{V_\beta} T)(\underset{1}{\omega}^H, ..., \underset{r}{\omega}^{V_2}, \overset{1}{X}^H, ..., \overset{s}{X}^{V_2}) = X^{V_\beta}(T(\underset{1}{\omega}^H, ..., \underset{r}{\omega}^{V_2}, \overset{1}{X}^H, ..., \overset{s}{X}^{V_2})-$$

$$-T(D_X^{V_\beta} \underset{1}{\omega}^H, ..., \underset{r}{\omega}^{V_2}, \overset{1}{X}^H, ..., \overset{s}{X}^{V_2}) - ... - T(\underset{1}{\omega}^H, ..., \underset{r}{\omega}^{V_2}, \overset{1}{X}^H, ..., D_X^{V_\beta} \overset{s}{X})$$

$$(\beta = 1, 2).$$

By a straightforward calculus, one obtains the local writing:

$$D_X^H T = X^{(0)m} T^{i_1...i_r}_{j_1...j_s|m} \delta_{i_1} \otimes ... \otimes \delta_{2i_r} \otimes dx^{j_1} \otimes ... \otimes \delta y^{(2)j_s},$$

where

$$T^{i_1...i_r}_{j_1...j_s|m} = \delta_m T^{i_1...i_r}_{j_1...j_s} + \underset{(00)}{L}{}^{i_1}_{hm} T^{hi_2...i_r}_{j_1...j_s} + ... + \underset{(20)}{L}{}^{i_r}_{hm} T^{i_1...i_{r-1}h}_{j_1...j_s} -$$
$$- \underset{(00)}{L}{}^{h}_{j_1 m} T^{i_1...i_r}_{hj_2...j_s} - ... - \underset{(20)}{L}{}^{h}_{j_s m} T^{i_1...i_r}_{j_1...j_{s-1}h}.$$

Similarly,

$$D_X^{V_\beta} T = X^{(\beta)m} T^{i_1...i_r}_{j_1...j_s} \overset{(\beta)}{|}_m \delta_{i_1} \otimes ... \otimes \delta_{2i_r} \otimes dx^{j_1} \otimes ... \otimes \delta y^{(2)j_s},$$

where

$$T^{i_1...i_r}_{j_1...j_s} \overset{(\beta)}{|}_m = \delta_{\beta m} T^{i_1...i_r}_{j_1...j_s} + \underset{(0\beta)}{C}{}^{i_1}_{hm} T^{hi_2...i_r}_{j_1...j_s} + ... + \underset{(2\beta)}{C}{}^{i_r}_{hm} T^{i_1...i_{r-1}h}_{j_1...j_s} -$$
$$- \underset{(0\beta)}{C}{}^{h}_{j_1 m} T^{i_1...i_r}_{hj_2...j_s} - ... - \underset{(2\beta)}{C}{}^{h}_{j_s m} T^{i_1...i_r}_{j_1...j_{s-1}h} \ (\beta = 1, 2).$$

4. d-TENSORS OF TORSION AND CURVATURE

The torsion

$$T(X, Y) = D_X Y - D_Y X - [X, Y]$$

of the N-linear connection D is well determined by its components which are d-tensors of $(1, 2)$-type ([1], [7], [8]):

$$v_\gamma T(\delta_{\beta k}, \delta_{\alpha j}) = \underset{(\alpha\beta)}{\overset{(\gamma)}{T}}{}^{i}_{jk} \delta_{\gamma i} \qquad (\alpha, \beta, \gamma = 1, 2).$$

In the notations in the cited papers, we have

$$hT(\delta_k,\delta_j) = \underset{(00)}{\overset{(0)}{T}}{}^i{}_{jk}\delta_i = \underset{(00)}{T}{}^i{}_{jk}\delta_i, \qquad v_\gamma T(\delta_k,\delta_j) = \underset{(00)}{\overset{(\gamma)}{T}}{}^i{}_{jk}\delta_{\gamma i} = \underset{(0\gamma)}{T}{}^i{}_{jk}\delta_{\gamma i},$$

$$hT(\delta_{\beta k},\delta_j) = \underset{(0\beta)}{\overset{(0)}{T}}{}^i{}_{jk}\delta_i = \underset{(\beta 0)}{P}{}^i{}_{jk}\delta_i, \qquad v_\gamma T(\delta_{\beta k},\delta_j) = \underset{(0\beta)}{\overset{(\gamma)}{T}}{}^i{}_{jk}\delta_{\gamma i} = \underset{(\beta\gamma)}{P}{}^i{}_{jk}\delta_{\gamma i},$$

$$v_\gamma T(\delta_{2k},\delta_{1j}) = \underset{(12)}{\overset{(\gamma)}{T}}{}^i{}_{jk}\delta_{\gamma i} = \underset{(2\gamma)}{Q}{}^i{}_{jk}\delta_{\gamma i}$$

$$v_\gamma T(\delta_{\beta k},\delta_{\beta j}) = \underset{(\beta\beta)}{\overset{(\gamma)}{T}}{}^i{}_{jk}\delta_{\gamma i} = \underset{(\beta\gamma)}{S}{}^i{}_{jk}\delta_{\gamma i}$$

$(\beta,\gamma = 1,2)$. The detailed expressions of $\underset{(\alpha\beta)}{\overset{(\gamma)}{T}}{}^i{}_{jk}$ $(\alpha,\beta,\gamma = 0,1,2)$ can be found in [1].

The curvature of the N-linear connection D,

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z,$$

is completely determined by its components (which are d-tensors)

$$R(\delta_{\gamma l},\delta_{\beta k})\delta_{\alpha j} = \underset{(\alpha\beta\gamma)}{R}{}_j{}^i{}_{kl}\delta_{\alpha i} \ (\alpha,\beta,\gamma = 0,1,2).$$

Namely, the 2-forms of curvature of an N-linear connection are, [1],

$$\underset{(\alpha)}{\Omega}{}^i{}_j = \frac{1}{2}\underset{(0\alpha)}{R}{}_j{}^i{}_{kl}dx^k\wedge dx^l + \underset{(1\alpha)}{P}{}_j{}^i{}_{kl}dx^k\wedge\delta y^{(1)l} + \underset{(2\alpha)}{P}{}_j{}^i{}_{kl}dx^k\wedge\delta y^{(2)l} +$$
$$\frac{1}{2}\underset{(1\alpha)}{S}{}_j{}^i{}_{kl}\delta y^{(1)k}\wedge\delta y^{(1)l} + \underset{(2\alpha)}{Q}{}_j{}^i{}_{kl}dy^{(1)k}\wedge\delta y^{(2)l} + \frac{1}{2}\underset{(2\alpha)}{S}{}_j{}^i{}_{kl}\delta y^{(2)k}\wedge\delta y^{(2)l}$$

$(\alpha = 0,1,2)$, where the coefficients $\underset{(0\alpha)}{R}{}_j{}^i{}_{kl}$, $\underset{(\beta\alpha)}{P}{}_j{}^i{}_{kl}$, $\underset{(\beta\alpha)}{Q}{}_j{}^i{}_{kl}$, $\underset{(\beta\alpha)}{S}{}_j{}^i{}_{kl}$ $(\alpha = 0,1,2;$ $\beta = 1,2)$ are d-tensors, named the **d-tensors of curvature** of the N-linear connection D. For a JN-linear connection, there holds

$$\underset{(0)}{\Omega}{}^i{}_j = \underset{(1)}{\Omega}{}^i{}_j = \underset{(2)}{\Omega}{}^i{}_j.$$

The detailed expressions of the d-tensors of curvature can be found in [1].

5. METRIC STRUCTURES ON T^2M

A **Riemannian metric** on T^2M is a tensor field G of type $(0,2)$, which is nondegenerate in each $u\in T^2M$ and is positively defined on T^2M.

In this paper, we shall consider metrics in the form

$$G = \underset{(0)}{g}{}_{ij}dx^i\otimes dx^j + \underset{(1)}{g}{}_{ij}\delta y^{(1)i}\otimes\delta y^{(1)j} + \underset{(2)}{g}{}_{ij}\delta y^{(2)i}\otimes\delta y^{(2)j}, \tag{10}$$

where $\underset{(\alpha)}{g}_{ij} = \underset{(\alpha)}{g}_{ij}(x, y^{(1)}, y^{(2)})$; this is, so that the distributions N, N_1 and V_2 generated by the nonlinear connection N be orthogonal with respect to G.

An N-linear connection D is called a **metrical** N-linear connection if $D_X G = 0$, $\forall X \in \mathcal{X}(T^2M)$, this is

$$\underset{(\alpha)}{g}_{ij|k} = \underset{(\alpha)}{g}_{ij} \overset{\beta}{|}_k = 0 \ (\alpha = 0,1,2;\ \beta = 1,2).$$

The existence of metrical N-linear connections is proved in [1]. Remember that a metrical JN-linear connection is the one used by R. Miron and Gh. Atanasiu in [7], namely $C\Gamma(N) = (L^i{}_{jk}, \underset{(1)}{C}{}^i{}_{jk}, \underset{(2)}{C}{}^i{}_{jk})$, given by

$$\begin{aligned} L^i{}_{jk} &= \frac{1}{2} g^{ih}\left(\frac{\delta g_{jh}}{\delta x^k} + \frac{\delta g_{hk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}\right), \\ \underset{(\beta)}{C}{}^i{}_{jk} &= \frac{1}{2} g^{ih}\left(\frac{\delta g_{jh}}{\delta y^{(\beta)k}} + \frac{\delta g_{hk}}{\delta y^{(\beta)j}} - \frac{\delta g_{jk}}{\delta y^{(\beta)h}}\right) \quad (\beta = 1,2), \end{aligned}$$

where $\underset{(0)}{g}_{ij} = \underset{(1)}{g}_{ij} = \underset{(2)}{g}_{ij} = g_{ij}$ (g_{ij} being a Riemannian metric on M) and g^{ij} are the elements of the inverse matrix of (g_{ij}).

6. MAXWELL EQUATIONS

Let T^2M be endowed with a nonlinear connection N, a Riemannian metric G and a metrical N-linear connection D.

Let $z^{(1)i}$, $z^{(2)i}$ the Liouville vector fields (7). We denote by

$$\overset{(\alpha)}{D}{}^i{}_j = z^{(\alpha)i}{}_{|j}, \qquad \overset{(\alpha\beta)}{d}{}^i{}_j = z^{(\alpha)i} \overset{(\beta)}{|}_j \quad (\alpha = 0,1,2;\ \beta = 1,2),$$

the **deflection tensor fields** of the N-linear connection D. By lowering and raising indices, we obtain the covariant deflection tensors

$$\overset{(\alpha)}{D}_{ij} = \underset{(\alpha)}{g}_{ih} \overset{(\alpha)}{D}{}^h{}_j, \qquad \overset{(\alpha\beta)}{d}_{ij} = \underset{(\alpha)}{g}_{ih} \overset{(\alpha\beta)}{d}{}^h{}_j \quad (\alpha = 0,1,2;\ \beta = 1,2),$$

and the contravariant deflection tensors

$$\overset{(\alpha)}{D}{}^{ij} = \underset{(\alpha)}{g}{}^{hj} \overset{(\alpha)}{D}{}^i{}_h, \qquad \overset{(\alpha\beta)}{d}{}^{ij} = \underset{(\alpha)}{g}{}^{hj} \overset{(\alpha\beta)}{d}{}^i{}_h \quad (\alpha = 0,1,2;\ \beta = 1,2).$$

By means of the deflection tensors constructed above, we can define the **electromagnetic tensor fields** by

$$\overset{(\alpha)}{F}_{ij} = \frac{1}{2}\left(\overset{(\alpha)}{D}_{ji} - \overset{(\alpha)}{D}_{ij}\right), \qquad \overset{(\alpha\beta)}{f}_{ij} = \frac{1}{2}\left(\overset{(\alpha\beta)}{d}_{ji} - \overset{(\alpha\beta)}{d}_{ij}\right)$$

$(\alpha = 0,1,2,\ \beta = 1,2)$.

In the particular case when the connection D is $C\Gamma(N)$, the electromagnetic tensors look as those in [7], that is,

$$\overset{(\alpha)}{F}_{ij}=\frac{1}{2}\left(\frac{\delta z^{(\alpha)}_j}{\delta x^i}-\frac{\delta z^{(\alpha)}_i}{\delta x^j}\right),\ \overset{(\alpha\beta)}{f}_{ij}=\frac{1}{2}\left(\frac{\delta z^{(\alpha)}_j}{\delta y^{(\beta)i}}-\frac{\delta z^{(\alpha)}_i}{\delta y^{(\beta)j}}\right)$$

$(\alpha=0,1,2,\ \beta=1,2)$.

The corresponding contravariant tensors are

$$\overset{(\alpha)}{F}{}^{ij}=\frac{1}{2}\left(\overset{(\alpha)}{D}{}^{ji}-\overset{(\alpha)}{D}{}^{ij}\right),\qquad \overset{(\alpha\beta)}{f}{}^{ij}=\frac{1}{2}\left(\overset{(\alpha\beta)}{d}{}^{ji}-\overset{(\alpha\beta)}{d}{}^{ij}\right),$$

or,

$$\text{(11)}\qquad \begin{aligned} 2\overset{(\alpha)}{F}{}^{ij} &= \underset{(\alpha)}{g}{}^{ih}z^{(\alpha)j}{}_{|h}-\underset{(\alpha)}{g}{}^{jh}z^{(\alpha)i}{}_{|h}\\ 2\overset{(\alpha\beta)}{f}{}^{ij} &= \underset{(\alpha)}{g}{}^{ih}z^{(\alpha)j}\overset{(\beta)}{|}_{h}-\underset{(\alpha)}{g}{}^{jh}z^{(\alpha)i}\overset{(\beta)}{|}_{h}\end{aligned}$$

$(\alpha=0,1,2,\ \beta=1,2)$.

By applying the Ricci identities (see [1]) of the N-linear connection D to the covariant electromagnetic tensor fields, there follows a generalization of the first Maxwell equations in the case of the 2-tangent bundle:

THEOREM 1. *The covariant electromagnetic tensors* $\overset{(\alpha)}{F}_{ij}$, $\overset{(\alpha\beta)}{f}_{ij}$ *satisfy the following identities:*

- $$2\{\overset{(\alpha)}{F}_{ji|k}+\overset{(\alpha)}{F}_{kj|i}+\overset{(\alpha)}{F}_{ik|j}\}=\sum_{(i,j,k)}\{\underset{(\alpha 00)}{R}{}_{hijk}z^{(\alpha)h}-\sum_{\delta=0}^{2}\underset{(00)}{\overset{(\delta)}{T}}{}^{m}_{jk}\overset{(\alpha\delta)}{d}{}_{im}\},$$

- $$2\{\overset{(\alpha)}{F}_{ji}\overset{(\beta)}{|}_{k}+\overset{(\alpha)}{F}_{kj}\overset{(\beta)}{|}_{i}+\overset{(\alpha)}{F}_{ik}\overset{(\beta)}{|}_{j}+\overset{(\alpha\beta)}{f}_{ji|k}+\overset{(\alpha\beta)}{f}_{kj|i}+\overset{(\alpha\beta)}{f}_{ik|j}\}=$$
$$=\sum_{(i,j,k)}\{(\underset{(\alpha 0\beta)}{R}{}_{hijk}-\underset{(\alpha 0\beta)}{R}{}_{hikj})z^{(\alpha)h}-\sum_{\delta=0}^{2}(\underset{(\beta 0)}{\overset{(\delta)}{T}}{}^{m}_{jk}-\underset{(\beta 0)}{\overset{(\delta)}{T}}{}^{m}_{kj})\overset{(\alpha\delta)}{d}{}_{im}\},$$

- $$2\{\overset{(\alpha\beta)}{f}_{ji}\overset{(\gamma)}{|}_{k}+\overset{(\alpha\beta)}{f}_{kj}\overset{(\gamma)}{|}_{i}+\overset{(\alpha\beta)}{f}_{ik}\overset{(\gamma)}{|}_{j}+\overset{(\alpha\gamma)}{f}_{ji}\overset{(\beta)}{|}_{k}+\overset{(\alpha\gamma)}{f}_{kj}\overset{(\beta)}{|}_{i}+\overset{(\alpha\gamma)}{f}_{ik}\overset{(\beta)}{|}_{j}\}=$$
$$=\sum_{(i,j,k)}\{(\underset{(\alpha\gamma\beta)}{R}{}_{hijk}-\underset{(\alpha\gamma\beta)}{R}{}_{hikj})z^{(\alpha)h}-\sum_{\delta=0}^{2}(\underset{(\beta\gamma)}{\overset{(\delta)}{T}}{}^{m}_{jk}-\underset{(\beta\gamma)}{\overset{(\delta)}{T}}{}^{m}_{kj})\overset{(\alpha\delta)}{d}{}_{im}\},$$

- $$2\{\overset{(\alpha\beta)}{f}_{ji}\overset{(\beta)}{|}_{k}+\overset{(\alpha\beta)}{f}_{kj}\overset{(\beta)}{|}_{i}+\overset{(\alpha\beta)}{f}_{ik}\overset{(\beta)}{|}_{j}\}=\sum_{(i,j,k)}\{\underset{(\alpha\beta\beta)}{R}{}_{hijk}z^{(\alpha)h}-\sum_{\delta=0}^{2}\underset{(\beta\beta)}{\overset{(\delta)}{T}}{}^{m}_{jk}\overset{(\alpha\delta)}{d}{}_{im}\}$$

($\alpha = 0, 1, 2,\ \beta = 1, 2$), where $\sum_{(i,j,k)}$ *means cyclic sum with respect to the indices* i, j, k.

In the particular case when D is the canonical JN-linear connection $C\Gamma(N)$, the relations above are identical to those given in [7].

In the following, by generalizing to T^2M the construction in [14], let us consider the vector fields $\overset{(\alpha\beta)}{J}$ given by their v_γ-components ($v_0 = h$):

(12)
$$v_\gamma \overset{(\alpha 0)}{J} = \left(\overset{(\alpha)}{F}{}^{ij} \overset{(\gamma)}{|}_j \right) \delta_{\gamma j}, \quad v_\gamma \overset{(\alpha\beta)}{J} = \left(\overset{(\alpha\beta)}{f}{}^{ij} \overset{(\gamma)}{|}_j \right) \delta_{\gamma j} \ (\alpha, \beta = 1, 2;\ \gamma = 0, 1, 2),$$

where in the right terms above there is no sum after γ.

The equalities 12 formally generalize the second Maxwell equations. We thus can call $\overset{(\alpha\beta)}{J}$, **current densities.**

We can obtain a generalization to T^2M of the charge conservation law by computing the divergence of $\overset{(\alpha\beta)}{J}$. More precisely, we have

THEOREM 2. *The following equalities hold:*

$$\begin{aligned} v_\gamma \overset{(\alpha 0)}{J}{}^i \overset{(\gamma)}{|}_i &= \frac{1}{2}\sum_{\gamma=0}^{2} \left\{ \left(\underset{(\gamma\gamma)}{\overset{1}{R}}{}_{ij} - \underset{(\gamma\gamma)}{\overset{1}{R}}{}_{ji} \right) \overset{(\alpha)}{F}{}^{ij} - \sum_{\delta=0}^{2} \underset{(\gamma\gamma)}{\overset{(\delta)}{T}}{}^m_{ij} \overset{(\alpha)}{F}{}^{ij} \overset{(\delta)}{|}_m \right\}, \\ v_\gamma \overset{(\alpha\beta)}{J}{}^i \overset{(\gamma)}{|}_i &= \frac{1}{2}\sum_{\gamma=0}^{2} \left\{ \left(\underset{(\gamma\gamma)}{\overset{1}{R}}{}_{ij} - \underset{(\gamma\gamma)}{\overset{1}{R}}{}_{ji} \right) \overset{(\alpha\beta)}{f}{}^{ij} - \sum_{\delta=0}^{2} \underset{(\gamma\gamma)}{\overset{(\delta)}{T}}{}^m_{ij} \overset{(\alpha\beta)}{f}{}^{ij} \overset{(\delta)}{|}_m \right\} \end{aligned}$$

($\alpha, \beta = 1, 2$), *where* $\underset{(\gamma\gamma)}{\overset{1}{R}}{}_{ij} = \sum_{\delta=0}^{2} \underset{(\delta\gamma\gamma)}{R}{}_i{}^m{}_{jm}$ ($\gamma = 0, 1, 2$) *are the Ricci tensors attached to* D, *and in the left terms above we mean sum after* γ (*and* i).

In the equations above, for each pair of distributions (α, β), the right terms play the role of the variation of the charge density ρ from the classical theory (up to a multiplication by -2).

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