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ON CERTAIN SUBCLASSES OF p-VALENTLY ANALYTIC FUNCTIONS OF ORDER α

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Abstract. The object of the present paper is to derive various properties and characteristics of certain subclasses of p-valently analytic functions of order α in the open unit disc by using the techniques involving the Briot-Bouquet differential subordination.
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1. INTRODUCTION AND DEFINITIONS

Let $A_p(n)$ denote the class of functions of the following form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k}\, z^{p+k} \ (p, n \in \mathbb{N} = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in \mathbb{C}$ and $|z| < 1\}$. We set $A_p(1) = A_p\,(p \in \mathbb{N})$. A function $f(z) \in A_p(n)$ is said to be in the class $S_p(n, \alpha)$ of p-valently starlike functions of order α if it satisfies the following inequality:

$$\operatorname{Re}\left\{\frac{z\, f'(z)}{f(z)}\right\} > \alpha \quad (0 \le \alpha < p;\ z \in U). \tag{1.2}$$

A function $f(z) \in A_p(n)$ is said to be in the class $K_p(n, \alpha)$ of p-valently convex functions of order α if it satisfies the following inequality:

$$\operatorname{Re}\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} > \alpha \quad (0 \le \alpha < p;\ z \in U). \tag{1.3}$$

It follows from (1.2) and (1.3) that

$$f(z) \in K_p(n, \alpha) \Leftrightarrow \frac{z\, f'(z)}{p} \in S_p^*(n, \alpha).$$

The classes $S_p(n, \alpha)$ and $K_p(n, \alpha)$ were studied by Aouf et al. [1], see [9] as well for more details. In particular, the class $S_p(1, \alpha) = S_p^*(\alpha)$ $(0 \le \alpha < p\,; p \in \mathbb{N})$ was considered by Patil and Thakare [10]. We also set $K_p(1, \alpha) = K_p(\alpha)$ $(0 \le \alpha < p\,;\, p \in N)$.

We now introduce an interesting subclasses of $A_p(n)$ as follows:

DEFINITION 1. A function $f(z)$ from the class $A_p(n)$ is said to be in the class $R_{p,j}(n,A,B,\alpha)$ if it satisfies the following subordination condition:

$$\frac{(p-j)!}{p!}\,\frac{f^{(j)}(z)}{z^{p-j}} \prec \frac{1+[B+(A-B)(1-\frac{(p-j)!}{p!}\,\alpha)]\,z}{1+B\,z} \tag{1.4}$$

$$\left(z\in U; 0\le j\le p, -1\le B<A\le 1, 0\le\alpha<\frac{p!}{(p-j)!}\right).$$

We note that:
(i) $R_{p,j}(n,A,B,0)=R_{p,j}(n,A,B)$ (Srivastava et al. [12]);
(ii) $R_{p,j}(n,1,-1,\alpha)=R_{p,j}(n,\alpha)$, where $R_{p,j}(n,\alpha)$ denotes the class of functions $f(z)\in A_p(n)$ satisfying the following inequality:

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\}>\alpha\ \left(0\le\alpha<\frac{p!}{(p-j)!}\ ;\ z\in U\ ;\ 0\le j\le p\right).$$

DEFINITION 2. A function $f(z)\in A_p$ is said to be in the class $H^{\lambda}_{p,j}(A,B,\alpha)$ if it satisfies the following subordination condition:

$$\begin{aligned}&(1-\lambda)\frac{z\,f^{(j)}(z)}{f^{(j-1)}(z)}+\lambda\left(1+\frac{z\,f^{(j+1)}(z)}{f^{(j)}(z)}\right)\\ &\prec(p-j+1)\frac{1+[B+(A-B)(1-\frac{\alpha}{p-j+1})]\ z}{B\ z}\end{aligned} \tag{1.5}$$

for some real number $\lambda, z\in U; 1\le j\le p; -1\le B<A\le 1,\ 0\le\alpha<p-j+1$).

We note that:
(i) $H^{\lambda}_{p,j}(A,B,0)=H^{\lambda}_{p,j}(A,B)$ (Srivastava et al. [12]);
(ii) $H^{\lambda}_{p,j}(1,-1,\alpha)=H^{\lambda}_{p,j}(\alpha)$, where $H^{\lambda}_{p,j}(\alpha)$ is the class of functions $f(z)\in A_p$ satisfying the following inequality:

$$\mathrm{Re}\left\{(1-\lambda)\,\frac{z\,f^{(j)}(z)}{f^{(j-1)}(z)}+\lambda\left(1+\frac{z\,f^{(j+1)}(z)}{f^{(j)}(z)}\right)\right\}>\alpha\ (z\in U) \tag{1.6}$$

for some real number λ, $1\le j\le p; 0\le\alpha<p-j+1$.

We also note that:

(1) $H^{\lambda}_{p,p}(\alpha)=H_p(\lambda,\alpha)$, where $H_p(\lambda,\alpha)$ is the class of functions $f(z)\in A_p$ satisfying the following inequality:

$$\mathrm{Re}\left\{(1-\lambda)\frac{z\,f^{(p)}(z)}{f^{(p-1)}(z)}+\lambda\left(1+\frac{z\,f^{(p+1)}(z)}{f^{(p)}(z)}\right)\right\}>\alpha\ (z\in U)$$

for some real number λ $(\lambda\ge 0), 0\le\alpha<1$.

(2) $H^{\lambda}_{p,1}(\alpha)=M_p(\lambda,\alpha)$, where $M_p(\lambda,\alpha)$ is the class of functions $f(z)\in A_p$ satisfying the following inequality:

$$\mathrm{Re}\left\{(1-\lambda)\,\frac{z\,f^{'}(z)}{f(z)}+\lambda(1+\frac{z\,f^{''}(z)}{f^{'}(z)})\right\}>\alpha$$

for some real number λ $(\lambda \geq 0), 0 \leq \alpha < p$. The class $M_p(\lambda, \alpha)$ of p-valently λ-convex of order α was studied by Owa [9].

We further set: $H^0_{p,j}(A, B, \alpha) = H_{p,j}(A, B, \alpha)$, $H^0_{p,j}(\alpha) = H_{p,j}(\alpha)$, $H^\lambda_{p,j}(0) = H^\lambda_{p,j}$, $H^\lambda_{p,1}(0) = M_p(\lambda)$ and $H^\lambda_{p,p}(0) = H_p(\lambda)$.

The class $H_p(\lambda)$ of p-valently λ-convex functions was introduced by Nunokawa [5] and was studied subsequently by Saitoh et al. [11] and Owa [9]. The class $M_p(\lambda)$ of p-valently Mocanu functions was investigated recently by Dziok and Stankiewicz [2] and Owa [8].

In the present paper, we derive various properties and characteristics of functions belonging to the classes $R_{p,j}(n, A, B, \alpha)$ and $H^\lambda_{p,j}(A, B, \alpha)$ by using the techniques involving the Briot–Bouquet differential subordinations.

2. PRELIMINARIES

In our present investigation of the classes $R_{p,j}(n, A, B, \alpha)$ and $H_{p,j}(A, B, \alpha)$, we require each of the following lemmas.

LEMMA 1. ([3]) *Let $h(z)$ be a convex (univalent) function in U such that $h(0) = 1$. Also let*

$$\varphi(z) = 1 + c_1\, z^n + c_2\, z^{n+1} + \ldots \tag{2.1}$$

be analytic in U. If

$$\varphi(z) + \frac{z\varphi^{'}(z)}{\gamma} \prec h(z) \quad (z \in U)$$

for some complex number $\gamma \neq 0$ with $\mathrm{Re}(\gamma) \geq 0$*, then*

$$\varphi(z) \prec \Psi(z) = \frac{\gamma}{n}\, z^{\frac{-\gamma}{n}} \int_0^z t^{\frac{\gamma}{n}-1} h(t)\mathrm{d}t \;\prec h(z) \quad (z \in U)$$

and $\Psi(z)$ is the best dominant.

LEMMA 2. ([4]) *If $-1 \leq B < A \leq 1, \beta \geq 0$, and* $\mathrm{Re}(\gamma) \geq -\frac{\beta(1-A)}{1-B}$*, then the following differential equation*

$$q(z) + \frac{z\,q^{'}(z)}{\beta\,q(z)+\gamma} = \frac{1+A\,z}{1+B\,z}$$

has a univalent solution in U given by

$$q(z)=\begin{cases}\dfrac{z^{\beta+\gamma}(1+Bz)^{\frac{\beta(A-B)}{B}}}{\beta\int\limits_0^z t^{\beta+\gamma-1}(1+Bt)^{\frac{\beta(A-B)}{B}}\mathrm{d}t}-\dfrac{\gamma}{\beta} & (B\neq 0),\\[2ex] \dfrac{z^{\beta+\gamma}\,\mathrm{e}^{\beta A z}}{\beta\int\limits_0^z t^{\beta+\gamma-1}\exp(\beta At)\mathrm{d}t}-\dfrac{\gamma}{\beta} & (B=0).\end{cases} \tag{2.2}$$

Furthermore, if φ is analytic in U and satisfies the following subordination condition:

$$\varphi(z)+\frac{z\,\varphi'(z)}{\beta\,\varphi(z)+\gamma}\prec\frac{1+A\,z}{1+B\,z}\quad(z\in U),$$

then

$$\varphi(z)\prec q(z)\prec\frac{1+A\,z}{1+B\,z}\quad(z\in U)$$

and $q(z)$ is the best dominant.

Lemma 3. ([14]) *Let μ be a positive measure on the unit interval $I=[0,1]$. Let $g(t,z)$ be a function analytic in U, for each $t\in I$, and integrable in t, for each $z\in U$ and for almost all $t\in I$. Suppose also that*

$$\mathrm{Re}\{g(t,z)\}>0\ \ (z\in U\ ;\quad t\in I),$$

$g(t,-r)$ is real for real r, and

$$\mathrm{Re}\left(\frac{1}{g(t,z)}\right)\geq\frac{1}{g(t,-r)}\quad(|z|\ \leq r<1;\ t\in I).$$

If

$$g(z)=\int_I g(t,z)\,\mathrm{d}\mu(t),$$

then

$$\mathrm{Re}\left(\frac{1}{g(z)}\right)\geq\frac{1}{g(-r)}\ (|z|\ \leq r<1).$$

For real or complex numbers $a,b,$ and c ($c\neq 0,-1,-2,\dots$), the hypergeometric function is defined by

$${}_2F_1(a,b;c;z)=1+\frac{a\cdot b}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\dots\ . \tag{2.3}$$

We note that the series in (2.3) converges absolutely for $z\in U$ and hence represents an analytic function in U.

Each of the identities (asserted by Lemma 4 below) is well known (cf. e.g., [13, Ch. 14]).

Lemma 4. *For real or complex numbers* a, b *and* c $(c \neq 0, -1, -2, \dots)$,

$$\int_0^1 t^b (1-t)^{c-b-1} (1-t\,z)^{-a} \mathrm{d}t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z)$$

$$(\mathrm{Re}(c) > \mathrm{Re}(b) > 0); \tag{2.4}$$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right); \tag{2.5}$$

$${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z); \tag{2.6}$$

$$(b+1){}_2F_1(1,b;b+1;z) = (b+1) + bz\ {}_2F_1(1,b+1;b+2\ ;\ z) \tag{2.7}$$

and

$${}_2F_1(a,b;\frac{a+b+1}{2};\frac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}\ . \tag{2.8}$$

Lemma 5. ([7]) *Let* $\varphi(z)$ *be analytic in* U *with* $\varphi(0) = 1$ *and* $\varphi(z) \neq 0$ $(0 < |z| < 1)$. *Also let*

$$\left|\frac{\nu(A-B)}{B} - 1\right| \leq 1 (-1 \leq B < A \leq 1;\ B \neq 0;\ \nu \in C \setminus \{0\})$$

or

$$\left|\frac{\nu(A-B)}{B} + 1\right| \leq 1 (-1 \leq B < A \leq 1;\ B \neq 0;\ \nu \in C \setminus \{0\}).$$

If $\varphi(z)$ *satisfies the following subordination condition*

$$1 + \frac{z\,\varphi'(z)}{\nu\ \varphi(z)} \prec \frac{1 + A\,z}{1 + B\,z}\ \ (z \in U), \tag{2.9}$$

$$\varphi(z) \prec \Psi(z) = (1 + B\,z)^{\frac{\nu(A-B)}{B}}\ \ (z \in U)$$

and $\Psi(z)$ *is the best dominant.*

3. MAIN RESULTS

Theorem 1. *Let* $-1 \leq B < A \leq 1$, $0 \leq j \leq p$, *and* $0 \leq \alpha < \frac{p!}{(p-j)!}$. *If* $f(z) \in R_{p,j}(n, A, B, \alpha)$, *then*

$$\begin{aligned} &\frac{(p-j)!}{p!}\ \frac{\mu+p}{z^{\mu+p}} \int_0^z t^{\mu+j-1} f^{(j)}(t)\mathrm{d}t \prec \widetilde{q}(z) \\ &\prec \frac{1 + [B + (A-B)(1 - \frac{(p-j)!}{p!}\ \alpha)]\,z}{1 + B\,z}\ (z \in U;\ 0 < \mu + p), \end{aligned} \tag{3.1}$$

where $\widetilde{q}(z)$*, given by*

$$(3.2)\qquad \widetilde{q}(z)=\begin{cases}\frac{[B+(A-B)(1-\frac{(p-j)!}{p!}\alpha)]}{B}+\frac{(B-A)\,(1-\frac{(p-j)!}{p!}\alpha)}{B}\,(1+B\,z)^{-1}\\ \quad\cdot\,{}_2F_1(1,1;\frac{\mu+p}{n}+1;\frac{B\,z}{B\,z+1})\qquad (B\neq 0),\\ 1+\frac{(\mu+p)[B+(A-B)(1-\frac{(p-j)!}{p!}\alpha)]z}{\mu+p+n}\qquad (B=0),\end{cases}$$

is the best dominant of (3.1).

Furthermore,

(3.3)

$$\operatorname{Re}\left\{\frac{\mu+p}{z^{\mu+p}}\int_0^z t^{\mu+j-1}f^{(j)}(t)\mathrm{d}t\right\}>\frac{p!}{(p-j)!}\,\rho(n,p,\mu,A,B,\alpha)\,(z\in U),$$

where

$$\rho(n,p,\mu,A,B,\alpha)=\begin{cases}\frac{[B+(A-B)(1-\frac{(p-j)!}{p!}\alpha)]}{B}+\frac{(B-A)(1-\frac{(p-j)!}{p!}\alpha)}{B}\\ \quad\cdot(1-B)^{-1}\,{}_2F_1(1,1;\frac{\mu+p}{n}+1;\frac{B}{B-1})\qquad (B\neq 0),\\ 1-\frac{(\mu+p)[B+(A-B)(1-\frac{(p-j)!}{p!}\alpha)]}{\mu+p+n}\qquad (B=0)\,.\end{cases}$$

The result is the best possible.

Proof. By setting

$$(3.4)\qquad \varphi(z)=\frac{(p-j)!}{p!}\,\frac{\mu+p}{z^{\mu+p}}\int_0^z t^{\mu+j-1}f^{(j)}(t)\mathrm{d}t\ (z\in U),$$

we note that $\varphi(z)$ is of the form (2.1) and analytic in U. On differentiating (3.4) with respect to z and simplifying, we get

$$\varphi(z)+\frac{z\,\varphi'(z)}{\mu+p}=\frac{(p-j)!}{p!}\,\frac{f^{(j)}(z)}{z^{p-j}}$$
$$\prec\frac{1+[B+(A-B)(1-\frac{(p-j)!}{p!}\alpha)]\,z}{1+B\,z}\ (z\in U).$$

Thus, by using Lemma 1 for $\nu=\mu+p$, we have

$$\begin{aligned}
&\frac{(p-j)!}{p!}\ \frac{\mu+p}{z^{\mu+p}}\int_0^z t^{\mu+j-1}f^{(j)}(t)\mathrm{d}t \prec \widetilde{q}(z)\\
=&\ \frac{\mu+p}{n}\ z^{-(\frac{\mu+p}{n})}\int_0^z t^{\frac{\mu+p-n}{n}}\ \frac{1+[B+(A-B)(1-\frac{(p-j)!}{p!}\ \alpha)]t}{1+B\,t}\ \mathrm{d}t\\
=&\ \begin{cases}\dfrac{[B+(A-B)(1-\frac{(p-j)!}{p!}\ \alpha)]}{B}+\dfrac{(B-A)\,(1-\frac{(p-j)!}{p!}\ \alpha)}{B}\ (1+B\ z)^{-1}\\[2mm] \qquad\cdot\ {}_2F_1(1,1;\frac{\mu+p}{n}+1;\frac{B\,z}{B\,z+1})\qquad (B\neq 0),\\[2mm] 1+\dfrac{(\mu+p)[B+(A-B)(1-\frac{(p-j)!}{p!}\ \alpha)]z}{\mu+p+n}\ \ (B=0),\end{cases}
\end{aligned}$$

by changing of variables followed by the use of the identities (2.4), (2.5), (2.6) and (2.7), successively. This proves assertion (3.1) of Theorem 1. Next, we show that

$$\inf_{|z|<1}\ \{\mathrm{Re}(\widetilde{q}(z))\}=\widetilde{q}(-1). \tag{3.5}$$

Indeed, for $|z|\le r<1$, we have

$$\begin{aligned}
&\mathrm{Re}\left(\frac{1+[B+(A-B)(1-\frac{(p-j)!}{p!}\ \alpha)]z}{1+B\ z}\right)\\
=&\ \mathrm{Re}\left\{(1-\frac{(p-j)!}{p!}\ \alpha)\frac{1+A\ z}{1+B\ z}+\frac{(p-j)!}{p!}\ \alpha\right\}\\
\ge&\ (1-\frac{(p-j)!}{p!}\ \alpha)\frac{1-Ar}{1-Br}+\frac{(p-j)!}{p!}\ \alpha\\
=&\ \frac{1-[B+(A-B)(1-\frac{(p-j)!}{p!}\ \alpha)]r}{1-B\ r}\qquad (\,|z|\le r<1).
\end{aligned}$$

Putting

$$G(s,z)=\frac{1+[B+(A-B)(1-\frac{(p-j)!}{p!}\ \alpha)]s\,z}{1+B\,s\ z}$$

$$\left(0\le s\le 1;\ 0\le\alpha<\frac{(p-j)!}{p!}\ ;\ \ z\in U\right)$$

and letting

$$\mathrm{d}\mu(s)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\ s^{a-1}(1-s)^{c-a-1}\mathrm{d}s\ ,$$

which is a positive measure on the closed interval $[0,1]$, $\widetilde{q}(z)$ can be rewritten as follows:

$$\widetilde{q}(z) = \int_0^1 G(s,z)\mathrm{d}\mu(s),$$

so that

$$\begin{aligned}\mathrm{Re}\{\widetilde{q}(z)\} &\geq \int_0^1 \frac{1-[B+(A-B)(1-\frac{(p-j)!}{p!}\ \alpha)]s\,r}{1-B\,s\,r}\,\mathrm{d}\mu(s)\\ &= \widetilde{q}(-r)(\,|z|\leq r<1),\end{aligned}$$

which, on letting $r \to 1^-$, yields (3.5). This proves (3.3). The estimate in (3.3) is the best possible as the function $\widetilde{q}(z)$ is the best dominant of (3.1). □

Putting $A = 1$ and $B = -1$ in Theorem 1, we obtain the following corollary:

COROLLARY 1. *Let* $0 \leq j \leq p$ *and* $0 \leq \alpha < \frac{(p-j)!}{p!}$. *If* $f(z) \in R_{p,j}(n,\alpha)$, *then*

$$\mathrm{Re}\left\{\frac{\mu+p}{z^{\mu+p}}\int_0^{!} zt^{\mu+j-1}f^{(j)}(t)\mathrm{d}t\right\} > \xi(n,p,\mu,\alpha)\ (z\in U),$$

where

$$\xi(n,p,\mu,\alpha) = \frac{(p-j)!\alpha}{p!} + \left(1-\frac{(p-j)!}{p!}\ \alpha\right)\left[{}_2F_1\left(1,1;\ 1+\frac{\mu+p}{n};\frac{1}{2}\right)-1\right].$$

The result is the best possible.

REMARK 1. (i) Corollary 1 improves the corresponding results of Aouf et al. [1], for $j = 0$ and $j = 1$, respectively.

(ii) Corollary 1 improves the corresponding result of Srivastava et al. [12].

(iii) For $A = 1, B = -1, n = p = 1$ and $j = 0$, Corollary 1 improves a result due to Obradovic [6].

THEOREM 2. *Let* $-1 \leq B < A \leq 1, 1 \leq j \leq p, 0 \leq \alpha < p-j+1$, *and* $\lambda > 0$. *If* $f(z) \in H^{\lambda}_{p,j}(A,B,\alpha)$, *then*

$$\frac{z\,f^{(j)}(z)}{(p-j+1)\,f^{(j-1)}(z)} \prec \widetilde{q}_1(z) = \frac{\lambda}{(p-j+1)Q(z)} \prec$$

$$\frac{1+[B+(A-B)(1-\frac{\alpha}{p-j+1}\ \alpha)]\,z}{1+B\,z}\ (z\in U), \tag{3.6}$$

where

$$Q(z) = \begin{cases} \int\limits_0^z t^{\frac{(p-j-\lambda+1)}{\lambda}}\left(\frac{1+Bt\,z}{1+B\,z}\right)^{\frac{(p-j+1)(A-B)(1-\frac{\alpha}{p-j+1})}{\lambda\,B}}\mathrm{d}t & (B\neq 0),\\ \int\limits_0^z t^{\frac{(p-j-\lambda+1)}{\lambda}}\ \exp\left(\frac{(p-j+1)(t-1)A(1-\frac{\alpha}{p-j+1})\,z}{\lambda}\right)\mathrm{d}t & (B=0),\end{cases} \tag{3.7}$$

and $\widetilde{q}_1(z)$ *is the best dominant of* (3.6). *Furthermore, if* $-1 \le B < 0$ *and* $A \le \dfrac{-\lambda B}{(p-j+1)(1-\frac{\alpha}{p-j+1})}$, *then* $f(z) \in H_{p,j}(\Im(p,j,A,B,\lambda,\alpha))$, *where*

$$\Im(p,j,A,B,\lambda,\alpha) =$$

$$(p-j+1)\left[{}_2F_1\left(1, \frac{(p-j+1)(B-A)(1-\frac{\alpha}{p-j+1})}{\lambda B}; \frac{p-j+1}{\lambda}+1; \frac{B}{B-1}\right)\right]^{-1}.$$

The result is the best possible.

Proof. Defining the function $\phi(z)$ by

$$\phi(z) = \frac{z\, f^{(j)}(z)}{(p-j+1)f^{(j-1)}(z)} \quad (1 \le j \le p;\ z \in U). \tag{3.8}$$

We note that

$$\varphi(z) = 1 + w_1 z + w_2 z^2 + \dots \tag{3.9}$$

is analytic in U. Making use of the logarithmic differentiation in (3.8) and using (1.5), we find that

$$\varphi(z) + \frac{\lambda\, z\, \varphi'(z)}{(p-j+1)\varphi(z)} \prec$$

$$\frac{1+[B+(A-B)(1-\frac{\alpha}{p-j+1})]\, z}{1+B\, z} \quad (z \in U). \tag{3.10}$$

Now, by using Lemma 2 for $\beta = \frac{p-j+1}{\lambda}$ and $\gamma = 0$, we obtain

$$\frac{z\, f^{(j)}(z)}{(p-j+1)f^{(j-1)}(z)} \prec \widetilde{q}_1(z) = \frac{\lambda}{(p-j+1)Q(z)}$$

$$\prec \quad \frac{1+[B+(A-B)(1-\frac{\alpha}{p-j+1})]\, z}{1+B\, z} \quad (z \in U),$$

where $\widetilde{q}_1(z)$ is the best dominant of (3.10) and $Q(z)$ is given by (3.7). Next, we show that

$$\inf_{|z|<1} \{\mathrm{Re}(\widetilde{q}(z))\} = \widetilde{q}(-1). \tag{3.11}$$

If we set

$$a = \frac{(p-j+1)(B-A)(1-\frac{\alpha}{p-j+1})}{\lambda\, B}, \ b = \frac{(p-j+1)}{\lambda}, \text{ and } c = b+1,$$

so that $c > b > 0$, then by using (2.4), (2.5), and (2.6), we find from (3.7) that

$$Q(z) = (1+Bz)^a \int_0^1 s^{b-1}(1+B\, s\, z)^{-a}\, \mathrm{d}s$$

$$= \frac{\Gamma(b)}{\Gamma(c)}\ {}_2F_1\left(1, a; c; \frac{B\ z}{1+B\ z}\right). \tag{3.12}$$

Since $B < 0$ and $A \leq \frac{-\lambda B}{(p-j+1)(1-\frac{\alpha}{p-j+1})}$, together, imply that $c > a > 0$, by using (2.4), (3.12) yields

$$Q(z) = \int_0^1 g(s,z)\ \mathrm{d}\mu(s),$$

where

$$g(s,z) = \frac{1+B\ z}{1+(1-s)B\ z} \quad \text{and} \quad \mathrm{d}\mu(s) = \frac{\Gamma(c)}{\Gamma(c)\Gamma(c-a)}\ s^{a-1}(1-s)^{c-a-1}\mathrm{d}s$$

is a positive measure on the closed interval $[0,1]$.

For $-1 \leq B < 1$, we note that $\mathrm{Re}\{g(s,z)\} > 0 (z \in U;\ s \in [0,1])$, $g(s,-r)$ is real, for $0 \leq r < 1$ and $s \in [0,1]$, and

$$\mathrm{Re}\left\{\frac{1}{g(s,z)}\right\} \geq \frac{1-(1-s)Br}{1-Br} = \frac{1}{g(s,-r)}$$
$$(|z| \leq r < 1;\ s \in [0,1]).$$

Therefore, by using Lemma 3, we have

$$\mathrm{Re}\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-r)} \quad (|z| \leq r < 1),$$

which, upon letting $r \to 1^-$, yields

$$\mathrm{Re}\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-1)} \quad .$$

In the case $A = -\frac{\lambda B}{(p-j+1)(1-\frac{\alpha}{p-j+1})}$, we obtain the required result by letting $A \to (-\frac{\lambda B}{(p-j+1)(1-\frac{\alpha}{p-j+1})})^+$ in $\widetilde{q}_1(z) = \frac{\lambda}{(p-j+1)Q(z)}$, where $Q(z)$ is given as above. The result is sharp because of the best dominant property of $\widetilde{q}_1(z)$. □

Remark 2. Putting (i) $\alpha = 0$, (ii) $A = 1$ and $B = -1$, (iii) $A = 1$, $B = -1$ and $\alpha = 0$, we obtain the results obtained by Srivastava et al. [12].

Theorem 3. *Let* $1 \leq j \leq p, \lambda \geq 0, \mu + p - j + 1 \geq 0, -1 \leq B < A \leq 1$, *and* $0 \leq \alpha < p - j + 1$, *such that*

$$B < A \leq 1 + \frac{\mu(1-B)}{p-j+1} \ .$$

(i) *If* $f(z) \in H^{\lambda}_{p,j}(A,B,\alpha)$, *then*

$$\frac{z^{\mu}\ f^{(j-1)}(z)}{(\mu+p-j+1)\int\limits_0^z t^{\mu-1}\ f^{(j-1)}(t)\mathrm{d}t} \prec \frac{1+A^*z}{1+B\,z} \quad (z \in U), \tag{3.13}$$

where

$$A^* = 1 - \frac{(p-j+1)(1-[B+(A-B)(1-\frac{\alpha}{p-j+1})]+\mu(1-B)}{\mu+p-j+1}.$$

Furthermore, if $f(z) \in H_{p,j}(A,B,\alpha)$, *then*

$$\frac{z^{\mu}\ f^{(j-1)}(z)}{(\mu+p-j+1)\int\limits_0^z t^{\mu-1}\ f^{(j-1)}(t)\mathrm{d}t} \prec \widetilde{q}_2(z) = \frac{1}{(\mu+p-j+1)Q(z)}$$

$$\prec \frac{1+[B+(A-B)(1-\frac{\alpha}{p-j+1})]z}{1+B\,z} \quad (z \in U), \tag{3.14}$$

where

$$Q(z) = \begin{cases} \int\limits_0^1 s^{\mu+p-j}\left(\frac{1+Bs\,z}{1+B\,z}\right)^{\frac{(p-j+1)(A-B)(1-\frac{\alpha}{p-j+1})}{B}} \mathrm{d}s & (B \neq 0), \\ \int\limits_0^1 s^{\mu+p-j} \exp((p-j+1)(s-1)A(1-\frac{\alpha}{p-j+1})z)\mathrm{d}s & (B = 0), \end{cases}$$

and $\widetilde{q}_2(z)$ *is the best dominant.*

(ii) *If* $-1 \leq B < 0$, $B < A \leq \min(1, \frac{\mu(1-B)}{p-j+1}, -\frac{(\mu+1)B}{p-j+1})$ *and* $f(z) \in H_{p,j}(A,B,\alpha)$, *then*

$$\mathrm{Re}\left(\frac{z^{\mu}\ f^{(j-1)}(z)}{\int\limits_0^z t^{\mu-1}\ f^{(j-1)}(t)\mathrm{d}t}\right) > \theta(p,j,\mu,A,B,\alpha) \quad (z \in U),$$

where

$$\theta(p,j,\mu,A,B,\alpha) = (\mu+p-j+1)$$
$$\left[{}_2F_1\left(1, \frac{(p-j+1)(B-A)(1-\frac{\alpha}{p-j+1})}{B};\ \mu+p-j+2; \frac{B}{B-1}\right)\right]^{-1}.$$

The result is the best possible.

Proof. By setting

$$\phi(z) = \frac{z^{\mu}\ f^{(j-1)}(z)}{(\mu+p-j+1)\int\limits_0^z t^{\mu-1}\ f^{(j-1)}(t)\mathrm{d}t} \quad (z \in U), \tag{3.14}$$

we see that $\phi(z)$ is of the form (3.9) and is analytic in U. On differentiating both sides of (3.14) followed by some obvious simplifications, we get

$$\frac{z\,f^{(j)}(z)}{(p-j+1)f^{(j-1)}(z)} = P(z) + \frac{zP'(z)}{(p-j+1)P(z)+\mu} \quad (z \in U),$$

where

$$P(z)=\frac{1}{p-j+1}\,[(\mu+p-j+1)\phi(z)-\mu]. \tag{3.15}$$

By using Lemma 2, we get

$$P(z)\prec q(z)\prec\frac{1+[B+(A-B)(1-\frac{\alpha}{p-j+1})]z}{1+B\,z}\ (z\in U), \tag{3.16}$$

where $q(z)$ is given by (2.2) with $\beta=p-j+1$ and $\gamma=\mu$. Again, by using (3.15) in (3.16), we get (3.13) and (3.14), respectively. The remaining part of the proof of Theorem 3 is similar to that of Theorem 2, and so we omit the details. □

REMARK 3. Putting (i) $\alpha=0$, (ii) $A=1$, $B=-1$ and $j=1$, (iii) $A=1$, $B=-1$, $j=1$ and $\mu=0$, we obtain the results obtained by Srivastava et al. [12].

Finally, we prove the following result:

THEOREM 4. *Let* $1\le j\le p,\ -1\le B<0,\ \lambda>0,\ B<A\le-\frac{\lambda B}{(p-j+1)(1-\frac{\alpha}{p-j+1})}$. *If* $f(z)\in H^{\lambda}_{p,j}(A,B,\alpha)$, *then, for* $z\in U$, *we have*

$$\left(\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right)^{\nu}\prec\left(\frac{p!}{(p-j+1)!}\right)^{\nu}(1-z)^{-2\nu[p-j-\Im(p,j,A,B,\lambda,\alpha)+1]} \tag{3.17}$$

where $\nu(p,j,A,B,\lambda,\alpha)$ *is defined as in Theorem 2 and* $\nu\neq 0$ *satisfies either*

$$|2\nu\,[(p-j+1)-\Im(p,j,A,B,\lambda,\alpha)]-1|\le 1$$

or

$$|2\nu\,[(p-j+1)-\Im(p,j,A,B,\lambda,\alpha)]+1|\le 1\ .$$

The result is the best possible.

Proof. Considering the function $\varphi(z)$ given by

$$\varphi(z)=\left(\frac{(p-j+1)!}{p!}\cdot\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right)^{\nu}\qquad(z\in U). \tag{3.18}$$

and choosing the principle branch in (3.18), we note that $\varphi(z)$ is of the form (3.9) and is analytic in U. On differentiating (3.18) logarithmically followed by the use of Theorem 2 in the resulting equation, we get

$$\begin{aligned}\frac{z\,f^{(j)}(z)}{(p-j+1)\,f^{(j-1)}(z)}&=1+\frac{z\,\varphi'(z)}{\nu(p-j+1)\varphi(z)}\\&\prec\frac{1+[1-\frac{2\Im(p,j,A,B,\lambda,\alpha)}{p-j+1}]\,z}{1-z}\qquad(z\in U),\end{aligned}$$

where $\Im(p,j,A,B,\lambda,\alpha)$ is defined us in Theorem 2. Now, the assertion of Theorem 4 follows by using Lemma 5. □

Putting $A = 1, B = -1$ and $j = 1$ in Theorem 4, we have the following corollary:

COROLLARY 2. *If* $f(z) \in M_p(\lambda, \alpha)$, *for* $\lambda \geq p - \alpha > 0$ *and* $0 \leq \alpha < p$, *then*

$$\left(\frac{f(z)}{z^p}\right)^{\nu} \prec (1-z)^{-2\nu[p-\Im(p,\lambda,\alpha)]} \quad (z \in U), \tag{3.19}$$

where

$$\Im(p, \lambda, \alpha) = \frac{p\Gamma(\frac{p}{\lambda} + \frac{1}{2})}{\sqrt{\pi}\ \Gamma(\frac{p-\alpha}{\lambda} + 1)}$$

and $\nu \neq 0$ *satisfies either*

$$|2\nu\left[(p - \Im(p, \lambda, \alpha)\right] + 1| \leq 1$$

or

$$|2\nu\left[(p - \Im(p, \lambda, \alpha)\right] - 1| \leq 1$$

The result is the best possible.

Putting $A = 1, B = -1$, and $j = p$ in Theorem 4 yields the following corollary:

COROLLARY 3. *If* $f(z) \in H_p(\lambda, \alpha), \lambda \geq 1 - \alpha$, *then*

$$\left(\frac{f^{(p-1)}(z)}{p!\ z}\right)^{\nu} \prec (1-z)^{-2\nu[1-\Im(\lambda,\alpha)]} \quad (z \in U), \tag{3.20}$$

where

$$\Im(\lambda, \alpha) = \frac{\Gamma(\frac{1-\alpha}{\lambda} + \frac{1}{2})}{\sqrt{\pi}\ \Gamma(\frac{1-\alpha}{\lambda} + 1)} \quad \textit{and} \quad \nu \neq 0$$

satisfies either

$$|2\nu\left[(1 - \Im(\lambda, \alpha)\right] + 1| \leq 1$$

or

$$|2\nu\left[(1 - \Im(\lambda, \alpha)\right] - 1| \leq 1\ .$$

The result is the best possible.

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