MATHEMATICA, Tome 49(72), N° 1, 2007, pp. 83–89

REGULAR SUBMODULES OF REGULAR KRONECKER MODULES

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Abstract. Using some results on the Hall algebra of the Kronecker algebra kK over the finite field k, we provide numerical criteria for a regular module in mod-kK to be embeddable in an another regular module. We also describe the possible factors of such an embedding. Finally the possible modules occurring in an n-term Hall product of regulars is discussed.

MSC 2000. 16G20.
Key words. Kronecker algebra, Hall algebra, regular modules, regular pencils.

1. INTRODUCTION

The Kronecker algebra kK, i.e. the path algebra over the Kronecker quiver

$$K:\ 1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2\ ,$$

is a special but important tame hereditary algebra, because in some sense models the behavior of all tame hereditary algebras. Its modules, called Kronecker modules, correspond to matrix pencils in linear algebra, so the Kronecker algebra relates representation theory with numerical linear algebra and matrix theory.

It is a natural question to find a necessary and sufficient condition (in terms of some numerical invariants called Kronecker invariants) for a Kronecker module to be isomorphic with the submodule of an another Kronecker module. This will help us in the description of the submodule category of the Kronecker algebra (see [8]) and also will lead us to the solution of the problem of giving necessary and sufficient conditions for the existence of a matrix pencil with prescribed Kronecker invariants and a prescribed arbitrary subpencil (see [4], [5]).

In this article we will study the embedding problem for regular Kronecker modules. Our approach will use results on the Hall algebra over the Kronecker algebra (and implicitly the classical Hall algebra).

2. KRONECKER MODULES AND THEIR INVARIANTS

Let K be the Kronecker quiver and k a finite field with $|k| = q$. We will consider the path algebra kK of K over k (called Kronecker algebra) and the category mod-kK of finite dimensional right modules over kK. The category mod-kK can and will be identified with the category rep-kK of the finite dimensional k-representations of the Kronecker quiver. Recall that such a representation is of the form

$$V_1 \overset{f}{\underset{g}{\leftleftarrows}} V_2 \ ,$$

where V_1, V_2 are finite dimensional k-spaces (corresponding to the two vertices) and $f, g : V_2 \to V_1$ are k-linear maps (corresponding to the two arrows). So up to isomorphism a Kronecker representation consists of two k-matrices of the same dimension. For general notions concerning the representation theory of quivers, we refer to [2], [7] or [1].

Up to isomorphism we will have two simple objects in mod-kK corresponding to the two vertices. We shall denote them by S_1 and S_2. For a module $M \in$ mod-kK, $[M]$ will denote the isomorphism class of M. The number of automorphisms of M will be denoted by α_M and the dimension vector of M by $\underline{\dim} M = (m_{S_1}(M), m_{S_2}(M))$, where $m_{S_i}(M)$ is the number of composition factors of M isomorphic to S_i. For a module M let $tM := M \oplus ... \oplus M$ (t-times).

For two modules $M, M' \in$ mod-kK we will denote by $M' \hookrightarrow M$ the fact that M' can be embedded in M (i.e. M' is isomorphic with a submodule of M) and by $M \twoheadrightarrow M'$ the fact that M projects on M' (i.e. M' is isomorphic with a factor module of M).

The indecomposables in mod-kK are divided into three families: the preprojectives, the regulars and the preinjectives.

The preprojective (respectively preinjective) indecomposable modules are up to isomorphism uniquely determined by their dimension vectors. For $n \in \mathbb{N}$ we will denote by P_n (respectively with I_n) the indecomposable preprojective module of dimension $(n+1, n)$ (respectively the indecomposable preinjective module of dimension $(n, n+1)$). So, P_0, P_1 are the projective indecomposable modules ($P_0 = S_1$ being simple) and $I_0 = S_2$, I_1 the injective indecomposable modules ($I_0 = S_2$ being simple).

The regular indecomposables (up to isomorphism) are $R_p(t)$ (for $t \geq 1$ and $p \in \mathbb{P}^1_k$) of dimension vector (td_p, td_p) (d_p standing for the degree of the point p).

For a partition $\lambda = (\lambda_1, ..., \lambda_s)$ we will use the notation $R_p(\lambda) = R_p(\lambda_1) \oplus ... \oplus R_p(\lambda_s)$.

Using the terminology of the Auslander-Reiten theory (see [2], [7] or [1]) the sequence

$$[R_p(1)], \ldots, [R_p(t)], \ldots$$

is the vertex-sequence of a standard homogeneous tube T_p. For this reason we say that the indecomposables $R_p(t)$ are taken from the tube T_p. We say that a module is taken from the tube T_p if all its indecomposable direct summands are isomorphic with some $R_p(t)$. So, the modules from T_p are isomorphic with $R_p(\lambda)$ for some partition λ.

The modules from a tube T_p also form a full, exact, extension-closed abelian subcategory of mod-kK with a single simple object: the regular indecomposable module on the mouth of the tube $R_p(1)$ (also called a quasi-simple module). A regular indecomposable module $R_p(t)$ is uniquely determined by its quasi-length t and quasi-socle $R_p(1)$, and it is quasi-uniserial, so it has a single quasi-composition series $0 \subset R_p(1) \subset R_p(2) \subset \cdots \subset R_p(t)$.

Note that $\mathrm{End}(R_p(1))$ is a field of index d_p over k and $\mathrm{End}(R_p(t))$ is a local k-algebra with $\dim_k \mathrm{End}(R_p(t)) = td_p$.

We also remark that we don't have nontrivial morphisms and extensions between two modules from different tubes, i.e. $\mathrm{Hom}(R_p(\lambda^p), R_{p'}(\lambda^{p'})) = \mathrm{Ext}^1(R_p(\lambda^p), R_{p'}(\lambda^{p'})) = 0$ for $p \neq p'$.

A module with all its indecomposable direct summands preprojective (resp. preinjective, regular) will be called preprojective (resp. preinjective, regular) module and denoted by P (resp. I, R). Using this notation it is well known that $\mathrm{Hom}(R,P) = \mathrm{Hom}(I,P) = \mathrm{Hom}(I,R) = \mathrm{Ext}^1(P,R) = \mathrm{Ext}^1(P,I) = \mathrm{Ext}^1(R,I) = 0$.

By Krull-Schmidt, every module in $M \in$ mod-kK (up to isomorphism) has the following decomposition:

$$(P_{c_1} \oplus \dots \oplus P_{c_n}) \oplus (\oplus_{p\in\mathbb{P}^1_k} R_p(\lambda^p)) \oplus (I_{d_1} \oplus \dots \oplus I_{d_m}),$$

where

(1) $(c_1, \dots, c_n)$ is a finite increasing sequence of nonnegative integers
(2) λ^p is a partition for every $p \in \mathbb{P}^1_k$
(3) $(d_1, \dots, d_m)$ is a finite decreasing sequence of nonnegative integers

The sequences from (1), (2), (3) will be called *Kronecker invariants* of the module M. We can see that they determine M up to isomorphism.

The defect of $M \in$ mod-kK with dimension vector (a,b) is defined in the Kronecker case as $\partial M := b - a$. Observe that if M is a preprojective (preinjective, respectively regular) indecomposable, then $\partial M = -1$ ($\partial M = 1$, respectively $\partial M = 0$). Moreover for a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in mod-kK we have $\partial M_2 = \partial M_1 + \partial M_3$.

3. THE HALL ALGEBRA APPROACH

Let k be a finite field. The Hall algebra $\mathcal{H}(kK, \mathbb{Q})$ associated to the Kronecker algebra kK is the free $\mathbb{Q}$-space having as basis the isomorphism classes in mod-kK together with a multiplication defined by:

$$[N_1][N_2] = \sum_{[M]} F^M_{N_1 N_2}[M],$$

where the structure constants

$$F^M_{N_1N_2} = |\{M \supseteq U|\ U \cong N_2,\ M/U \cong N_1\}|$$

are called Hall numbers. Notice that $\mathcal{H}(kK, \mathbb{Q})$ is an associative, usually non-commutative algebra with unit element the isoclass of the zero module.

More generally for $M, N_1, ..., N_t \in$ mod-kK we can define

$$F^M_{N_1...N_t} = |\{M = M_0 \supseteq M_1 \supseteq ... \supseteq M_t = 0|\ M_{i-1}/M_i \cong N_i, \forall 1 \leq i \leq t\}|.$$

We then have (using associativity)

$$[N_1]...[N_t] = \sum_{[M]} F^M_{N_1...N_t}[M].$$

If $F^M_{N_1...N_t} \neq 0$ then we will use the notation $[M] \in \{[N_1]...[N_t]\}$ and call $[M]$ a term in $[N_1]...[N_t]$, $\{[N_1]...[N_t]\}$ denoting the set of all terms in $[N_1]...[N_t]$.

The following lemma follows immediately from the definitions above.

Lemma 3.1. a) $\{[N_1][N_2]\} = \{[M]|F^M_{N_1N_2} \neq 0\} =$
$= \{[M]|$ *exists a short exact sequence* $0 \to N_2 \to M \to N_1 \to 0\}$.
b) $M' \hookrightarrow M \Leftrightarrow F^M_{XM'} \neq 0$ *for some* $X \Leftrightarrow [M] \in \{[X][M']\}$ *for some* X.
c) $M \twoheadrightarrow M' \Leftrightarrow F^M_{M'X} \neq 0$ *for some* $X \Leftrightarrow [M] \in \{[M'][X]\}$ *for some* X.

The lemma above shows that in order to characterize when a module is embeddable in an another one we can use our knowledge on Hall products.

Since we are interested in the regular Kronecker modules we would need information on their Hall products.

Using the remarks from Section 1, we notice that for $p \neq p'$

$$[R_p(\lambda^p)][R_{p'}(\lambda^{p'})] = [R_p(\lambda^p) \oplus R_{p'}(\lambda^{p'})].$$

This means that it is enough to consider regulars from the same tube.

For a tube T_p we will denote by $\mathcal{H}(T_p, \mathbb{Q})$ the unital subalgebra of $\mathcal{H}(kK, \mathbb{Q})$ generated by the classes $[R_p(t)]$ with $t \geq 1$. Observe that this algebra has as $\mathbb{Q}$-basis the classes $R_p(\lambda)$ with λ a partition.

It is well known that the algebra $\mathcal{H}(T_p, \mathbb{Q})$ coincides with the classical Hall algebra studied by P. Hall, so we can apply all the results due to Hall, Macdonald and Zelevinsky, getting all the information we need for the Hall product of regulars.

We will summarize in the following some of the notions, properties and facts related to the classical Hall algebras (see [6] II+Appendix for all the details).

First some notions related to partitions. $\mathbb{P}$ will denote the set of all partitions, $\mathbb{P}(n)$ the set of partitions of n. For $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{P}$ denote by $l(\lambda)$ the length, with $|\lambda|$ the weight and with λ' the conjugate (or transpose) of λ (i.e. $l(\lambda) = s$, $|\lambda| = \lambda_1 + \cdots + \lambda_s$). Let $m_i(\lambda)$ be the multiplicity of i in λ and $n(\lambda) = \sum(i-1)\lambda_i$.

For $\lambda, \mu \in \mathbb{P}$ we define the following so-called dominance ordering

$$\mu \trianglelefteq \lambda \Leftrightarrow |\mu| = |\lambda| \text{ and } \mu_1 + ... + \mu_i \leq \lambda_1 + ... + \lambda_i \text{ for all } i.$$

We shall write $\mu \subseteq \lambda$ to mean that $\mu_i \leq \lambda_i$ for all $i \geq 1$. For $\mu \subseteq \lambda$ we say that $\lambda - \mu$ is a horizontal t-strip if $|\lambda - \mu| := |\lambda| - |\mu| = t$ and the sequences λ and μ are interlaced, in the sense that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \ .$$

A tableau T is a sequence of partitions

$$0 = \lambda^0 \subseteq \lambda^1 \subseteq ... \subseteq \lambda^r = \lambda$$

such that $\lambda^i - \lambda^{i-1}$ is a horizontal strip for $i = \overline{1, r}$. The partition λ is called the shape of the tableau T and the sequence $(|\lambda^1 - \lambda^0|, ..., |\lambda^r - \lambda^{r-1}|)$ the weight of T. It is known that for given partitions λ, μ (with $|\lambda| = |\mu|$) there is a tableau of shape λ and weight μ iff $\mu \trianglelefteq \lambda$ (see [3]).

Theorem 3.2. (Hall, Zelevinsky) a) $\mathcal{H}(T_p, \mathbb{Q})$ *is commutative.*

b) *For partitions* $\lambda, \mu, \nu \in \mathbb{P}$ *there is a so called classical Hall polynomial* $g^\lambda_{\mu\nu} \in \mathbb{Z}[q]$ *independent from* k *and* p *such that* $F^{R_p(\lambda)}_{R_p(\mu)R_p(\nu)} = g^\lambda_{\mu\nu}(q^{d_p})$. *Moreover* $g^\lambda_{\mu\nu} = g^\lambda_{\nu\mu}$ *and* $g^\lambda_{\mu\nu} = 0$ *unless* $|\lambda| = |\mu| + |\nu|$ *and* $\mu, \nu \subseteq \lambda$.

c) $g^\lambda_{\mu(t)} = 0$ *unless* $\sigma = \lambda - \mu$ *is a horizontal* t *strip. If* $\sigma = \lambda - \mu$ *is a horizontal* t *strip then*

$$g^\lambda_{\mu(t)} = \begin{cases} \frac{q^{n(\lambda)-n(\mu)}}{1-q^{-1}} \prod_{i \in I}(1 - q^{-m_i(\lambda)}) & \text{for } t > 0 \\ 1 & \text{for } t = 0. \end{cases}$$

Denote by $c^\lambda_{\mu\nu}$ the leading coefficient of the classical Hall polynomial $g^\lambda_{\mu\nu}$. This is called Littlewood-Richardson coefficient and plays a crucial role in partition combinatorics.

4. REGULAR SUBMODULES OF REGULAR KRONECKER MODULES

It follows from the remarks in the previous sections that for $p_1, ..., p_s \in \mathbb{P}^1_k$ pairwise different and $\mu^{p_1}, ...\mu^{p_s}, \lambda^{p_1}, ..., \lambda^{p_s} \in \mathbb{P}$ we have that

$$R_{p_1}(\mu^{p_1}) \oplus ... \oplus R_{p_s}(\mu^{p_s}) \hookrightarrow R_{p_1}(\lambda^{p_1}) \oplus ... \oplus R_{p_s}(\lambda^{p_s}) \text{ iff}$$

$$R_{p_1}(\mu^{p_1}) \hookrightarrow R_{p_1}(\lambda^{p_1}) \ ... \ R_{p_s}(\mu^{p_s}) \hookrightarrow R_{p_s}(\lambda^{p_s}).$$

This means that it is enough to study the embedding problem on a single tube.

Theorem 4.1. *For two partitions* $\mu, \lambda \in \mathbb{P}$ *and* $p \in \mathbb{P}^1_k$ *we have*

$$R_p(\mu) \hookrightarrow R_p(\lambda) \text{ iff } \mu \subseteq \lambda.$$

Proof. "$\Rightarrow$" If $R_p(\mu) \hookrightarrow R_p(\lambda)$ then by Lemma 3.1. b) there is an $X \in$ mod-kK such that $F^{R_p(\lambda)}_{XR_p(\mu)} \neq 0$. We have then an exact sequence $0 \to R_p(\mu) \to R_p(\lambda) \to X \to 0$, so the defect $\partial X = 0$, which means that X is regular (since otherwise it must contain both preinjective and preprojective components, and we cannot have projection from a regular to a preprojective). Notice also that X must be from the tube T_p (this because we can't have nonzero morphisms between modules from different tubes), so there is a partition $\nu \in \mathbb{P}$ such that $X \cong R_p(\nu)$. Using Theorem 3.2. b), the classical Hall polynomial $g^{\lambda}_{\nu\mu} \neq 0$, so $|\mu| + |\nu| = |\lambda|$ and $\mu, \nu \subseteq \lambda$.

"$\Leftarrow$" If $\mu \subseteq \lambda$, there is a sequence of partitions

$$\mu = \mu^0 \subseteq \mu^1 \subseteq ... \subseteq \mu^{|\lambda-\mu|} = \lambda$$

such that $|\mu^{i+1} - \mu^i| = 1$.

Then $\mu^{i+1} - \mu^i$ is a horizontal 1-strip, so by Theorem 3.2. c) $g^{\mu^{i+1}}_{(1)\mu^i} \neq 0$, which by Lemma 3.1. means that $[R_p(\mu^{i+1})] \in \{[R_p(1)][R_p(\mu^i)]\}$. So by associativity in the Hall algebra

$$[R_p(\lambda)] \in \{[R_p(1)]^{|\lambda-\mu|}[R_p(\mu)]\},$$

which implies $R_p(\mu) \hookrightarrow R_p(\lambda)$ by Lemma 3.1. □

Remark 4.2. Using Lemma 3.1. a) it follows that $R_p(\mu) \hookrightarrow R_p(\lambda)$ with factor $R_p(\lambda)/R_p(\mu) \cong R_p(\nu)$ iff the Littlewood-Richardson coefficient $c^{\lambda}_{\nu\mu} \neq 0$.

Finally we will investigate the modules which will appear as terms in the Hall product $[R_p(\mu_s)]...[R_p(\mu_1)]$, i.e. we will describe the set

$$\{[R_p(\mu_s)]...[R_p(\mu_1)]\}$$

where $\mu_1 \geq ... \geq \mu_s \in \mathbb{N}^*$. Consider the partition $\mu = (\mu_1, ..., \mu_s)$.

Theorem 4.3. *We have*

$$\{[R_p(\mu_s)]...[R_p(\mu_1)]\} = \{[R_p(\lambda)] | \mu \trianglelefteq \lambda\}$$

Proof. We will apply Theorem 3.2. c) inductively and obtain that the terms in the Hall product above are of the form $[R_p(\lambda)]$, where λ is such that there is a tableau of shape λ and weight μ. This implies the desired result. □

Acknowledgements. The author thanks to grant CEEX ET19/2005 and to the Institute for Research Programs (KPI) of the Sapientia Foundation for support through a group research grant in the period 2005/2006.

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Received August 04, 2006

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