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MARX-STROHHACKER INEQUALITY FOR MOCANU-JANOWSKI α-CONVEX FUNCTIONS

Y. POLATOĞLU and E. YAVUZ

Abstract. Let Ω be the class of functions $w(z)$ regular in the unit disc $D = \{z : |z| < 1\}$ with $w(0) = 0$, and $|w(z)| < 1$. For arbitrarily fixed real numbers $A \in (-1, 1]$ and $B \in [-1, A)$, let $P(A,\ B)$ be the class of regular functions $p(z)$ in D such that $p(0) = 1$, and $p(z) \in P(A,\ B)$ if and only if $p(z) = \frac{1+Aw(z)}{1+Bw(z)}$ for every $z \in D$, for some $w(z) \in \Omega$.

In the present paper we apply the subordination principle to give new proofs for some results concerning the class $M(\alpha, A,\ B)$ of functions $f(z)$ regular in D with $f(0) = 0,\ f'(0) = 1$ satisfying the condition: $M(\alpha, A,\ B)$ if and only if $\left[(1-\alpha)z\frac{f'(z)}{f(z)} + \alpha\left(1 + z\frac{f''(z)}{f'(z)}\right)\right] = p(z)$, for all z in D and for some $p(z) \in P(A,\ B)$ $(A \in (-1, 1], B \in [-1, A), 0 \le \alpha < 1)$.

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1. INTRODUCTION

Let Ω be the family of functions $w(z)$ regular in the unit disc D and satisfying the conditions $w(0) = 0,\ |w(z)| < 1$, for $z \in D$.

For arbitrary fixed numbers $A,\ B,\ \ -1 \le B < A \le 1,$ let $P(A,\ B)$ denote the family of functions

$$p(z) = 1 + p_1z + p_2z^2 + ... + p_nz^n + ...$$

regular in D and such that $p(z)$ is in $P(A,\ B)$ if and only if

$$p(z) \prec \frac{1+Az}{1+Bz} \Leftrightarrow p(z) = \frac{1+Aw(z)}{1+Bw(z)}$$

for some $w(z) \in \Omega$ and every $z \in D$.

Furthermore, for arbitrary fixed numbers $A,\ B,\ \alpha,\ -1 \le B < A \le 1,\ 0 \le \alpha < 1$, let $M(\alpha, A,\ B)$ denote the family of functions

$$f(z) = z + a_2z^2 + a_3z^3 + ...$$

regular in D and such that $f(z)$ is in $M(\alpha, A,\ B)$ if and only if

$$\left[(1-\alpha)z\frac{f'(z)}{f(z)} + \alpha\left(1 + z\frac{f''(z)}{f'(z)}\right)\right] = p(z)$$

for some $p(z)$ in $P(A,\ B)$ and for all z in D.

Furthermore, for $A = 1,\ B = -1$, the class $M(\alpha, 1, -1)$ becomes the well known class of α-convex functions introduced by P.T. Mocanu ([4]).

2. NEW RESULTS ON THE CLASS $M(\alpha, A,\ B)$

In this section we shall give representation theorems and a generalized Marx-Strohhacker inequality for the class $M(\alpha, A,\ B)$. Our proofs are based on I.S. Jack's Lemma

LEMMA 1. ([3]) *Let $w(z)$ be a non-constant and analytic function in the unit disc D with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then $z_0 w'(z_0) = kw(z_0)$ and $k \geq 1$.*

THEOREM 1. *If $f(z)$ satisfies*

$$\left[(1-\alpha)z\frac{f'(z)}{f(z)} + \alpha\left(1 + z\frac{f''(z)}{f'(z)}\right) - 1\right] \prec \begin{cases} \frac{(A-B)z}{1+Bz} = F_1(z), & B \neq 0, \\ Az = F_2(z), & B = 0, \end{cases} \tag{1}$$

then $f(z) \in M(\alpha, A,\ B)$.

Proof. The linear transformation

$$w_1 = h(z) = \frac{(A-B)z}{1+Bz}$$

maps $|z| = r$ onto the circle centered at $C_1(r)$, with radius $\rho_1(r)$, where

$$\begin{cases} C_1(r) = -\frac{B(A-B)r^2}{1-B^2r^2},\ \rho_1(r) = \frac{(A-B)r}{1-B^2r^2}, & B \neq 0, \\ C_1(r) = (0,0), \qquad \rho_1(r) = |A|r, & B = 0. \end{cases}$$

Therefore $h(D)$ is contained in the closed disc centered at $C_1(r)$ with radius $\rho_1(r)$. On the other hand, we define the function $w(z)$ by

$$\frac{f(z)}{z}\left(z\frac{f'(z)}{f(z)}\right)^{\alpha} = \begin{cases} (1 + Bw(z))^{\frac{A-B}{B}}, & B \neq 0, \\ e^{Aw(z)}, & B = 0, \end{cases} \tag{2}$$

where $(1+Bw(z))^{\frac{A-B}{B}}$ and $e^{Aw(z)}$ have the value 1 at the origin. Then $w(z)$ is analytic in D, and $w(0) = 0$. If we take the logarithmic derivative of equality (2), simple calculations yield

$$(1-\alpha)z\frac{f'(z)}{f(z)} + \alpha\left(1 + z\frac{f''(z)}{f'(z)}\right) - 1 = \begin{cases} \frac{(A-B)zw'(z)}{1+Bw(z)}, & B \neq 0, \\ Azw'(z), & B = 0. \end{cases} \tag{3}$$

Now, the subordination (1) is equivalent to $|w(z)| < 1$ for all $z \in D$. Indeed, assume the contrary. There exists $z_0 \in D$, $Max_{|z|=z_0}$ such that $|w(z_0)| = 1$. Then, by I.S. Jack's lemma, $z_0 w'(z_0) = kw(z_0)$ and $k \geq 1$, and for such $z_0 \in D$ we have

$$(1-\alpha)z_0\frac{f'(z_0)}{f(z_0)} + \alpha\left(1 + z_0\frac{f''(z_0)}{f'(z_0)}\right) - 1 = \begin{cases} \frac{(A-B)kw(z_0)}{1+Bw(z_0)} = F_1(w(z_0)) \notin F_1(D), \\ \qquad B \neq 0, \\ Akw(z_0) = F_2(w(z_0)) \notin F_2(D), \\ \qquad B = 0, \end{cases}$$

since $|w(z_0)| = 1$ and $k \geq 1$. But this contradicts (3) and so $|w(z)| < 1$ for all $z \in D$.

On the other hand we have,

$$\left[(1-\alpha)z\frac{f'(z)}{f(z)} + \alpha\left(1 + z\frac{f''(z)}{f'(z)}\right)\right] - 1 \prec \begin{cases} \frac{(A-B)z}{1+Bz}, & B \neq 0, \\ Az, & B = 0, \end{cases} \Leftrightarrow$$

$$\left[(1-\alpha)z\frac{f'(z)}{f(z)} + \alpha\left(1 + z\frac{f''(z)}{f'(z)}\right)\right] - 1 = \begin{cases} \frac{(A-B)w(z)}{1+Bw(z)}, & B \neq 0, \\ Aw(z), & B = 0. \end{cases}$$

The sharpness of the result follows from the fact that

$$\frac{f(z)}{z}\left(z\frac{f'(z)}{f(z)}\right)^{\alpha} = \begin{cases} (1+Bz)^{\frac{A-B}{B}}, & B \neq 0, \\ e^{Az}, & B = 0, \end{cases}$$

implies

$$\left[(1-\alpha)z\frac{f'(z)}{f(z)} + \alpha\left(1 + z\frac{f''(z)}{f'(z)}\right)\right] - 1 \prec \begin{cases} \frac{(A-B)z}{1+Bz}, & B \neq 0, \\ Az, & B = 0. \end{cases}$$

□

Remark 1. If $f(z) \in M(\alpha, A,\ B)$, then

(4) $$w(z) = \begin{cases} \frac{1}{B}\left[\left(\frac{f(z)}{z}\right)^{\frac{B}{A-B}}\left(z\frac{f'(z)}{f(z)}\right)^{\frac{\alpha B}{A-B}} - 1\right], & B \neq 0, \\ \log\left[\left(\frac{f(z)}{z}\right)^{\frac{1}{A}}\left(z\frac{f'(z)}{f(z)}\right)^{\frac{\alpha}{A}}\right], & B = 0. \end{cases}$$

If we use the definition of $w(z)$, we get

$$\begin{cases} \left|\left(\frac{f(z)}{z}\right)^{\frac{B}{A-B}}\left(z\frac{f'(z)}{f(z)}\right)^{\frac{\alpha B}{A-B}} - 1\right| < |B|, & B \neq 0, \\ \left|\log\left[\left(\frac{f(z)}{z}\right)\left(z\frac{f'(z)}{f(z)}\right)^{\alpha}\right]\right| < |A|, & B = 0. \end{cases}$$

Substituting specific values for A, B and α, the following are obtained:

(1) $\alpha = 1, A = 1, B = -1$,

$$\left|\frac{1}{\sqrt{f'(z)}} - 1\right| < 1.$$

This is the well-known "Marx-Strohhacker Inequality" ([2, p. 129]) for convex functions.

(2) $\alpha = 0, A = 1, B = -1$,

$$\left|\sqrt{\frac{z}{f(z)}} - 1\right| < 1.$$

This inequality was proved by Marx-Strohhacker in 1932 and by M.S. Robertson in 1936 for starlike functions ([2, p. 128]).

(3) $A = 1, B = -1,$

$$\left|\sqrt{\frac{z}{f(z)}\left(\frac{f(z)}{zf'(z)}\right)^{\alpha}} - 1\right| < 1. \tag{5}$$

This is the Marx-Strohhacker inequality for α-convex functions.

(4) $A = 1, B = 0,$

$$\left|\log\left[\left(\frac{f(z)}{z}\right)\left(z\frac{f'(z)}{f(z)}\right)^{\alpha}\right] - 1\right| < 1.$$

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TC İstanbul Kültür University
Faculty of Science and Letters
Department of Mathematics and Computer Sciences
Bakırköy, 34156, İstanbul, Turkey
E-mail: y.polatoglu@iku.edu.tr
E-mail: e.yavuz@iku.edu.tr