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SOME INTEGRAL OPERATORS DEFINED ON p-VALENT FUNCTIONS BY USING HYPERGEOMETRIC FUNCTIONS

A. TEHRANCHI and S. R. KULKARNI

Abstract. In the present paper we introduce some integral operators and verify the effect of these operators on p-valent functions and find radii of starlikeness and convexity for these operators, finally we introduce the concept of neighborhood.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ be class of functions $f(z)$ of the form

$$f(z) = mz^p + \sum_{n=p-1}^{2p-1} t_{n-p+1} z^{n-p+1} - {}_2F_1(a,b;c;z),\ |z| < 1,$$

where ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} z^n$, $c > b > 0$, $c > a+b$, $(a,n) = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1,n-1)$, $t_{n-p+1} = \frac{(a,n-p+1)(b,n-p+1)}{(c,n-p+1)(n-p+1)!}$ and

$$m = \frac{\Gamma(c)\Gamma(a+p)\Gamma(b+p) + \Gamma(a)\Gamma(b)\Gamma(c+p)\Gamma(n+1)}{\Gamma(a)\Gamma(b)\Gamma(c+p)\Gamma(n+1)}.$$

These functions are analytic in the punctured unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For more details on hypergeometric functions ${}_2F_1(a,b;c;z)$ see [4] and [7].

Let $f \in \mathcal{A}$, we denote by UCV^p the class of uniformly convex p-valent function in Δ and $\alpha - ST$ the class of α - starlike functions also denote by $\alpha - UCV^p$ the class of α-uniformly convex p-valent function in Δ which are introduced and investigated by Kanas, Wiśniwoska [6] and Silverman [10] for $p = 1$.

The function $f(z)$ in $\mathcal{A}$ can be expressed in the form

$$f(z) = z^p - \sum_{n=p+1}^{\infty} k_n z^n,\ p \in \mathbb{N} \tag{1}$$

such that $k_n = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(n+1)}$, $n \geq p+1$.

Definition 1. Let $f \in \mathcal{A}$ and $0 \leq \alpha < \infty$. Then $f \in \alpha - UCV^p$ if and only if $\mathrm{Re}\left\{p + \frac{zf''}{f'}\right\} > \alpha \left|\frac{zf''}{f'}\right|$, $z \in \Delta$.

Definition 2. Let $f \in \mathcal{A}$. The class α - uniformly starlike functions $\alpha - US\mathcal{A}^p$ is defined as

$$\alpha - US\mathcal{A}^p = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'}{f}\right) > \alpha \left|\frac{zf'}{f} - p\right|, \ \alpha \geq 0, \quad z \in \Delta \right\}$$

Definition 3. (see [7], [11] and [12]). Let the function f be of the form (1) and be analytic in Δ. The fractional derivative of f of order δ is defined by

$$D_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\delta} d\xi, \ 0 \leq \delta < 1 \tag{2}$$

where the multiplicity of $(z-\xi)^\delta$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$ and so we have

$$D_z^\delta f(z) = \frac{1}{\Gamma(2-\delta)} z^{p-\delta} - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n+p-\delta)} \ a_n z^{n-\delta}. \tag{3}$$

Making use of (2) and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [11] introduced the operator

$$\Omega_z^\delta f(z) := \Gamma(2-\delta) z^\delta D_z^\delta f(z), \quad 0 \leq \delta < 1 \tag{4}$$

and for $\delta = 0$ we have $\Omega_z^0 f(z) = f(z)$.

Definition 4. Let $f(z) \in \mathcal{A}$ is said to be a member of the $\alpha - UCV_\delta^p(n, \phi)$ if $f(z)$ satisfies the inequality

$$\begin{aligned} &\mathrm{Re}\left(\frac{z(\Omega_z^\delta f(z))' + \eta z^2 (\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f) + \eta z (\Omega_z^\delta f(z))'}\right) \\ &\geq \alpha \left|\frac{z(\Omega_z^\delta f(z))' + z^2(\Omega_z^\delta f)''}{(1-\eta)\Omega_z^\delta f(z) + \eta z(\Omega_z^\delta f(z))'} - 1\right| + \sin\phi, \end{aligned} \tag{5}$$

where $0 \leq \eta \leq 1, 0 \leq \phi < \frac{\pi}{2}, p \in \mathbb{N}, \alpha \geq 0$ and $0 \leq \delta < 1$.

We note that by specializing the parameters $\alpha, \phi, \eta, \delta$ we obtain the following subclasses studied by various authors (by putting $\sin\phi = \beta$ and $f(z) = z^p - \sum_{n=0}^{\infty} a_n z^n, a_n \geq 0$).

(I) If $\beta = 0, \delta = 0$ and $p = 1 \Rightarrow \alpha - UCV(\alpha, 0) \equiv p_1(1, \lambda, \beta)$ was studied by Altintas [1].

(II) If $\eta = 0, \delta = 0, \alpha = 0, p = 1 \Rightarrow \alpha - UCV(0, \phi) \equiv T^*(\beta)$ was studied by Silverman [10].

(III) If $\eta = 1, \delta = 0, \alpha = 0, p = 1 \Rightarrow \alpha - UCV(1, \phi) \equiv C(\beta)$ was studied by Silverman [10].

(IV) If $\eta = 0, \delta =, 0, p = 1 \Rightarrow \alpha - UCV(0, \phi) \equiv UCT(k, \beta)$ was studied by R. Bharati, R. Parvatham and A. Swaminathan [5].

(V) If $p = 1, \eta = 0$ and $\beta = 0$ and $\delta = 0$, that is $k - sT$ introduced by Kanas and Wiśniowska [6].

(VI) If $\eta = 1, \beta = 0$ and $\delta = 0, p = 1$ that is $\alpha - UCV$ introduced and studied by Kanas and Wiśniowska [6].
(VII) If $p = 1, \delta = 0$ that is $\alpha - UCV(\eta, \beta)$ introduced and studied by E. Aqlan [3].

Remark 1. $\alpha - S\mathcal{A}^p \subset \alpha - UCV_\delta^p(\eta, \beta)$ when $\eta = 0$ and $\beta = 0$ and $\alpha - UCV^p \subset \alpha - UCV_\delta^p(\eta, \beta)$ when $\eta = 1, \beta = 0$.

Lemma 1. (Coefficient Bound) [13] *The function $f(z)$ defined by* (1) *is in the class $\alpha - UCV_\delta^p(\eta, \phi)$ if and only if*

$$\sum_{n=p+1}^{\infty} \gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi))]k_n \\ \leq (p-\sin\phi)(1-\eta+\eta p)+\alpha(p-1)(1-\eta) \tag{6}$$

where $\gamma^p(n,\delta) = \frac{\Gamma(2-\delta)\Gamma(n+p)}{\Gamma(n+p-\delta)}$ and $0 \leq \phi < \frac{\pi}{2}, \alpha \geq 0, 0 \leq \eta \leq 1, p \in \mathbb{N}$ and $0 \leq \delta < 1$.

2. SPECIAL FUNCTIONS AND INTEGRAL OPERATORS ON $\alpha - UCV_\delta^P(\eta, \phi)$

Definition 5. Let c be a real number such that $c > -p$. For $f \in \alpha - UCV_\delta^p(\eta, \phi)$, we define F_c by

$$F_c(z) = \frac{c+p}{z^c}\int_0^z s^{c-1} f(s)\mathrm{d}s \tag{7}$$

Theorem 1. *$F_c(z)$ defined by* (7) *belongs to $\alpha - UCV_\delta^p(\eta, \phi)$.*

Proof. Let $f(z) = z^p - \sum\limits_{n=p+1}^{\infty} k_n z^n \in \alpha - UCV_\delta^p(\eta, \phi)$ then

$$F_c(z) = \frac{c+p}{z^c}\int_0^z \left(s^{c-1+p} - \sum_{n=p+1}^{\infty} k_n s^{n+c-1}\right)\mathrm{d}s = z^p - \sum_{n=p+1}^{\infty}\frac{c+p}{n+c}k_n z^n.$$

Hence $F_c(z) = z^p - \sum\limits_{n=p+1}^{\infty}\frac{c+p}{c+n}k_n z^n$.

Therefore

$$\sum_{n=p+1}^{\infty} \gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi))]\frac{c+p}{c+n}k_n \\ \leq \sum_{n=p+1}^{\infty} \gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi))]k_n \\ \leq (p-\sin\phi)(1-\eta+\eta p)+\alpha(p-1)(1-\eta) \quad \text{(by 6).} \tag{8}$$

So $F_c(z) \in \alpha - UCV_\delta^p(\eta, \phi)$. □

Theorem 2. *The function $F_c(z)$ defined in 5 is starlike of order λ $(0 \leq \lambda < p)$ in $|z| < r_1(\eta, \phi, \alpha, \delta, n, p, c, \lambda)$, where*

$$r_1(\eta, \phi, \alpha, \delta, n, p, c, \lambda) = \inf_{n \geq p+1} \left\{ \frac{[(1-\eta+, n\eta)(n(1+\alpha) - (\alpha + \sin\phi)]}{(p - \sin\phi)(1-\eta+\eta p) + \alpha(p-1)(1-\eta)} \left(\frac{c+n}{c+p}\right)\left(\frac{p-\lambda}{2p-n-\lambda}\right)\gamma^p(n,\delta)\right\}^{\frac{1}{n-p}}.$$

The bound for $|z|$ is sharp for each n with extremal function being of the form

$$f_n(z) = z^p - \frac{(p-\sin\phi)(1-\eta+\eta p) + \alpha(p-1)(1-\eta)}{\gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha) - (\alpha+\sin\phi)]}\ z^n, n \geq p+1.$$

Proof. We must show that

$$\left|\frac{zF_c'(z)}{F_c(z)} - p\right| < p - \lambda \tag{9}$$

But we have

$$\left|\frac{zF_c'(z)}{F_c(z)} - p\right| \leq \frac{\sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n (p-n)|z|^{n-p}}{1 - \sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n |z|^{n-p}}.$$

Therefore (9) holds if $\sum_{n=p+1}^{\infty}\left(\frac{c+p}{c+n}\right)\left(\frac{2p-n-\lambda}{p-\lambda}\right) k_n|z|^{n-p} < 1$. Now in view of (8) the last inequality holds if

$$|z|^{n-p} < \frac{[(1-\eta+n\eta)(n(1+\alpha) - (\alpha+\sin\phi)]}{(p-\sin\phi)(1-\eta+\eta p) + \alpha(p-1)(1-\eta)} \left(\frac{p-\lambda}{2p-n-\lambda}\right)\left(\frac{c+n}{c+p}\right)\gamma^p(n,\delta).$$

This gives the required result. □

Corollary 1. *The function $F_c(z)$ defined in 5 is convex of order λ $(0 \leq \lambda < p)$ in $|z| < r_2(\eta, \phi, \alpha, \delta, n, p, c, \lambda)$, where*

$$r_2(\eta, \phi, \alpha, \delta, n, p, c, \lambda) = \inf_{n \geq p+1} \left\{ \frac{(1-\eta+n\eta)(n(1+\alpha) - (\alpha + \sin\phi))}{(p - \sin\phi)(1-\eta+\eta p) + \alpha(p-1)(1-\eta)} \left(\frac{p-\lambda}{2p-n-\lambda}\right)\left(\frac{c+n}{c+p}\right)\gamma^p(n,\delta)\right\}^{\frac{1}{n-p}}. \tag{10}$$

Proof. We must show that $\left|\frac{zF_c''(z)}{F_C'(z)}\right| < p - \lambda$ for $|z| < r_2$ and $c > -p$.
But we have

$$\left|\frac{zF_c''(z)}{F_c'(z)}\right| \leq \frac{p(p-1) + \sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n n(n-1)|z|^{n-p}}{p - \sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n n|z|^{n-p}}.$$

Therefore (10) holds if $\sum_{n=p+1}^{\infty} \frac{n(n-1+p-\lambda)}{p(2-p)-\lambda} \left(\frac{c+p}{c+n}\right) k_n |z|^{n-p} < 1$. Now in view of (10) the last inequality holds if

$$|z|^{n-p} < \frac{[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi)]}{(p-\sin\phi)(1-\eta+\eta p)+\alpha(p-1)(1-\eta)}$$
$$\left(\frac{p(2-p)-\lambda}{n(n-1+p-\lambda)}\right)\left(\frac{c+n}{c+p}\right)\gamma^p(n,\delta).$$

This gives the required result. □

Definition 6. Let c be a real number such that $c > -p$. Let $f \in \alpha - UCV_\delta^p(\eta,\phi)$, Komato operator in [8] is defined (for $p = 1$) by

$$G(z) = \int_0^1 \frac{(c+1)^\xi}{\Gamma(\xi)} t^c (\log\frac{1}{t})^{\xi-1} \frac{f(tz)}{t^p} \mathrm{d}t \;\; c > -1, \xi \geq 0.$$

Theorem 3. *$G(z)$ defined in 6 belongs to $\alpha - UCV_\delta^p(\eta,\phi)$.*

Proof. Since $\int_0^1 t^c(-\log t)^{\xi-1}\mathrm{d}t = \frac{\Gamma(\xi)}{(c+1)^\xi}$ and $\int_0^1 t^{n+c-p}(-\log t)^{\xi-1}\mathrm{d}t$ $= \frac{\Gamma(\xi)}{(c+n-p+1)^\xi}$ $n \geq p+1$. Therefore we obtain

$$\begin{aligned} G(z) &= \frac{(c+1)^\xi}{\Gamma(\xi)}\left[\int_0^1 t^c z^p \log(\frac{1}{t})^{\xi-1}\mathrm{d}t - \sum_{n=p+1}^{\infty}\int_0^t \log(\frac{1}{t})^{\xi-1} t^{n-p+c} k_n z^n \mathrm{d}t\right] \\ &= z^p - \sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^\xi k_n z^n \end{aligned} \tag{11}$$

Therefore and with use of (6) we have

$$\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi))]\left(\frac{c+1}{c+n-p+1}\right)^\xi k_n$$
$$\leq (p-\sin\phi)(1-\eta+\eta p)+\alpha(p-1)(1-\eta) \tag{12}$$

So $G(z) \in \alpha - UCV_\delta^p(\eta,\phi)$. □

Theorem 4. *The function $G(z)$ defined in 6 is starlike of order λ $(0 \leq \lambda < p)$ in $|z| < r_1(\eta,\phi,\alpha,\delta,n,p,c,\xi,\lambda)$ where*

$$r_1 = \inf_{n\geq p+1}\left\{\frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi)}{(p-\sin\phi)(1-\eta+\eta p)+\alpha(p-1)(1-\eta)}\right.$$
$$\left.\left(\frac{p-\lambda}{2p-n-\lambda}\right)\left(\frac{c+n-p+1}{c+1}\right)^\xi\gamma^p(n,\delta)\right\}^{\frac{1}{n-p}} \tag{13}$$

Proof. We must show that

$$\left|\frac{zG'(t)}{G(t)} - p\right| < p-\lambda. \tag{14}$$

By (11) we have

$$\left|\frac{zG'(t)}{G(t)}-p\right| \le \frac{\sum\limits_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}(p-n)k_n|z|^{n-p}}{1-\sum\limits_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}k_n|z|^{n-p}}.$$

Therefore (14) holds if $\sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}\frac{(2p-(n+\lambda))}{p-\lambda}k_n|z|^{n-p} < 1$. Now in view of (11) the last inequality holds if

$$|z|^{n-p} \le \frac{\gamma^p(n,\delta)(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi))}{(p-\sin\phi)(1-\eta+\eta p)+\alpha(p-1)(1-\eta)}$$
$$\left(\frac{p-\lambda}{(2p-(n+\lambda))}\right)\left(\frac{c+n-p+1}{c+1}\right)^{\xi}.$$

This gives the required result. □

Corollary 2. *The function $G(z)$ defined in 6 is convex of order λ $(0\le\lambda<p)$ in $|z|<r_2(\eta,\phi,\alpha,\delta,n,p,c,\xi,\lambda)$, where*

$$r_2 = \inf_{n\ge p+1}\left\{\frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi)}{(p-\sin\phi)(1-\eta+\eta p)+\alpha(p-1)(1-\eta)}\right.$$
$$\left.\left(\frac{c+n-p+1}{c+1}\right)^{\xi}\left(\frac{p(1-\lambda)}{n(p+n-\lambda-1)}\right)\gamma^p(n,\delta)\right\}^{\frac{1}{n-p}}.$$

Proof. We must show that

$$\left|\frac{zG''(z)}{G'(z)}\right| < p-\lambda, \qquad |z|<r_2. \tag{15}$$

By (11) we must show that

$$\left|\frac{p(p-1)z^{p-1}-\sum\limits_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}k_n n(n-1)|z|^{n-1}}{pz^{p-1}-\sum\limits_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}k_n|z|^{n-1}}\right| < p-\lambda.$$

Therefore

$$\sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}\left(\frac{n(p-\lambda+n-1)}{p(1-\lambda)}\right)k_n|z|^{n-p}<1. \tag{16}$$

Therefore (16) holds if

$$|z|^{n-p} < \frac{\gamma^p(n,\delta)(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi))}{(p-\sin\phi)(1-\eta+\eta p)+\alpha(p-1)(1-\eta)}$$
$$\left(\frac{c+n-p+1}{c+1}\right)^{\xi}\left(\frac{p(1-\lambda)}{n(p+n-\lambda-1)}\right).$$

□

Definition 7. Let $f \in \alpha - UCV_{\delta}^{p}(\eta, \phi)$. Function $H_{\mu}(z)$ defined by

$$H_{\mu}(z) = (1-\mu)z^{p} + \mu p \int_{0}^{z} \frac{f(t)}{t} \mathrm{d}t, \qquad \mu \geq 0, z \in \Delta.$$

Theorem 5. *The function $H_{\mu}(z)$ defined in 7 belongs to $\alpha - UCV_{\delta}^{p}(\eta, \phi)$ if $0 \leq \mu \leq p+1$.*

Proof. Let $f(z) \in \alpha - UCV_{\delta}^{p}(\eta, \phi)$ and is of the form (1) so

$$H_{\mu}(z) = z^{p} - \mu z^{p} + \mu p (\int_{0}^{z} \left(t^{p-1} - \sum_{n=p+1}^{\infty} k_{n} t^{n-1} \right) \mathrm{d}t) = z^{p} - \sum_{n=p+1}^{\infty} \left(\frac{\mu p}{n} k_{n} \right) z^{n} \tag{17}$$

Therefore we have by (6)

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} \gamma^{p}(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi))]\frac{\mu p}{n} k_{n} \\
&\leq \sum_{n=p+1}^{\infty} \gamma^{p}(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi))]\frac{\mu p}{p+1} k_{n} \\
&\leq \sum_{n=p+1}^{\infty} \gamma^{p}(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi))] k_{n} \\
&\leq (p-\sin\phi)(1-\eta+\eta p) + \alpha(p-1)(1-\eta).
\end{aligned}$$

So $H_{\mu}(z) \in \alpha - UCV_{\delta}^{p}(\eta, \phi)$. □

Remark 2. By the similar method which we applied for theorem 4 we obtain the radii of starlikeness and convexity of order λ $(0 \leq \lambda \leq p)$ for $H_{\mu}(z)$ respectively as following

$$\begin{aligned}
r_{1} &= \inf_{n \geq p+1} \left\{ \frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi))\gamma^{p}(n,\delta)}{(p-\sin\phi)(1-\eta+\eta p)+\alpha(p-1)(1-\eta)} \right. \\
&\quad \left. \left(\frac{p-\lambda}{2p-n-\lambda} \right) \left(\frac{n}{\mu p} \right) \right\}^{\frac{1}{n-p}}, \\
r_{2} &= \inf_{n \geq p+1} \left\{ \frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\sin\phi))\gamma^{p}(n,\delta)}{(p-\sin\phi)(1-\eta+\eta p)+\alpha(p-1)(1-\eta)} \right. \\
&\quad \left. \left(\frac{(1-\lambda)}{\mu(p+n-\lambda-1)} \right) \right\}^{\frac{1}{n-p}},
\end{aligned}$$

where $0 \leq \mu \leq p+1$.

3. (n, λ)-NEIGHBORHOOD

DEFINITION 8. (cf. [9]) Let $\lambda \geq 0$ and $f(z) \in \mathcal{A}$ and f defined by (1). Define the (n, λ)-neighborhood of a function $f(z)$ by

$$N_{n,\lambda}(f) = \left\{ g \in \mathcal{A} : g(z) = z^p - \sum_{n=p+1}^{\infty} k'_n z^n; \ \sum_{n=p+1}^{\infty} n|k_n - k'_n| \leq \lambda \right\}. \quad (18)$$

For the identity function $e(z) = z$, we have

$$N_{n,\lambda}(e) = \left\{ g \in \mathcal{A} : g(z) = z^p - \sum_{n=p+1}^{\infty} k'_n z^n \text{ and } \sum_{n=p+1}^{\infty} n|k'_n| \leq \lambda \right\}. \quad (19)$$

THEOREM 6. *Let*

$$\lambda = \frac{(p+1)(p - \sin\phi)(1 - \eta + \eta p) + \alpha(p-1)(1-\eta)}{\gamma^p(p+1,\delta)(1+p\eta)(p(1+\alpha) + 1 - \sin\phi)}.$$

where $\gamma^p(p+1,\delta) = \dfrac{\Gamma(2-\delta)\Gamma(2p+1)}{\Gamma(2p-\delta)}$.
Then

$$\alpha - UCV_\delta^p(\eta, \phi) \subset N_{n,\lambda}(e).$$

Proof. For $f \in \alpha - UCV_\delta^p(\eta, \phi)$ we have from (6)

$$\begin{aligned} &\sum_{n=p+1}^{\infty} (1+p\eta)(p(1+\alpha) + 1 - \sin\phi)]\gamma^p(p+1,\delta)k_n \\ &\leq \sum_{n=p+1}^{\infty} [(1-\eta+n\eta)(n(1+\alpha) - \alpha + \sin\phi)]\gamma^p(n,\delta)k_n \\ &\leq (p - \sin\phi)(1-\eta+\eta p) + \alpha(p-1)(1-\eta). \end{aligned}$$

Therefore

$$\sum_{n=p+1}^{\infty} k_n \leq \frac{(p - \sin\phi)(1-\eta+\eta p) + \alpha(p-1)(1-\eta)}{\gamma^p(p+1,\delta)(1+p\eta)(p(1+\alpha) + 1 - \sin\phi)}, \quad (20)$$

and on the other hand we have for $|z| < r$

$$\begin{aligned} |f'(z)| &\leq p|z|^{p-1} + |z|^p \sum_{n=p+1}^{\infty} nk_n \leq pr^{p-1} + r^p \sum_{n=p+1}^{\infty} nk_n \\ &\leq pr^{p-1} + r^p \frac{(p+1)(p-\sin\phi)(1-\eta+\eta p) + \alpha(p-1)(1-\eta)}{\gamma^p(p+1,\delta)(1+p\eta)(p(1+\alpha)+1-\sin\phi)} \quad \text{(from (20))}. \end{aligned}$$

From above inequalities we conclude

$$\sum_{n=p+1}^{\infty} nk_n \leq \frac{(p+1)(p-\sin\phi)(1-\eta+\eta p) + \alpha(p-1)(1-\eta)}{\gamma^p(p+1,\delta)(1+p\eta)(p(1+\alpha)+1-\sin\phi)} = \lambda.$$

$\square$

REMARK 3. Special case of theorem 6 when (i) $\alpha = 0, \eta = 0, p = 1, \delta = 0$ was proved recently by Altintas and Owa [2], (ii) for $p = 1, \delta = 0$ and with putting $\sin\phi = \beta$ we get a region that E. Aqlan has defined and studied in [3].

DEFINITION 9. The function $f(z)$ defined by (1) is said to be a member of the class $\alpha - UCV_\delta^{p,\xi}(\eta, \phi)$ if there exists a function $g \in \alpha - UCV_\delta^p(\eta, \phi)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| \le p - \xi,\ z \in \Delta,\ 0 \le \xi < p.$$

THEOREM 7. *If $g \in \alpha - UCV_\delta^p(\eta, \phi)$ and*

$$\xi = p - \frac{\lambda}{p+1}\mu(\eta, \phi, \alpha, \delta, p) \tag{21}$$

such that

$$\begin{aligned} &\mu(\eta, \phi, \alpha, \delta, p) \\ = \ &[\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\sin\phi)]/[\gamma^p(p+1,\delta)(1+p\eta) \\ &(p(\alpha+1)+1-\sin\phi) - (p-\sin\phi)(1-\eta+\eta p) + \alpha(p-1)(1-\eta)], \end{aligned}$$

then $N_{n,\lambda}(g) \subset \alpha - UCV_\delta^{p,\xi}(\eta, \phi)$.

Proof. Let $f \in N_{n,\lambda}(g)$, then we have from (18) that $\sum\limits_{n=p+1}^{\infty} n|k_n - k'_n| \le \lambda$ which readily implies the coefficient inequality

$$\sum_{n=p+1}^{\infty} |k_n - k'_n| \le \frac{\lambda}{p+1}.$$

Also since $g \in \alpha - UCV_\delta^p(\eta, \phi)$ we have from (6)

$$\sum_{n=p+1}^{\infty} k'_n \le \frac{(p-\sin\phi)(1-\eta+\eta p) + \alpha(p-1)(1-\eta)}{\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\sin\phi)}$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| = \left|\frac{z^p - \sum\limits_{n=p+1}^{\infty} k_n z^n - z^p + \sum\limits_{n=p+1}^{\infty} k'_n z^n}{z^p - \sum\limits_{n=p+1}^{\infty} k'_n z^n}\right| < \frac{\sum\limits_{n=p+1}^{\infty} |k_n - k'_n|}{1 - \sum_{n=p+1}^{\infty} k'_n}$$

$$\begin{aligned} &\le \left(\frac{\lambda}{p+1}\right)(\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\sin\phi)/ \\ &\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\sin\phi) - (p-\sin\phi)(1-\eta+\eta p) \\ &+ \alpha(p-1)(1-\eta)) = \left(\frac{\lambda}{p+1}\right)\mu(\eta, \phi, \alpha, \delta, p) = p - \xi. \end{aligned}$$

Then $\left|\frac{f(Z)}{g(z)} - 1\right| < p - \xi$. Thus, by Definition 9, $f \in \alpha - UCV_{\delta}^{p,\xi}(\eta, \phi)$ for ξ given by (21). □

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Department of Mathematics,
Fergusson College, University of Pune,
Pune - 411004, India
E-mail: kulkarni_ferg@yahoo.com
E-mail: Tehranchiab@yahoo.co.uk