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A UNIFIED TREATMENT OF CERTAIN UNIFORMLY ANALYTIC FUNCTIONS

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Abstract. In this paper we introduce and study some properties of a unified class $\mathcal{U}[\Phi, \Psi; \alpha, \beta, \lambda, n]$ of certain uniformly analytic functions with negative coefficients in a unit disk U. These properties include growth and distortion, radii of convexity, radii of starlikeness and radii of close-to-convexity. Further, results on integral transform are also given.

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1. INTRODUCTION

Denote by A the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$. Denote by $S^*(\alpha)$ the class of starlike functions $f \in A$ of order $\alpha(0 \leq \alpha < 1)$ satifying

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U$$

and let $C(\alpha)$ be the class of convex functions $f \in A$ of order $\alpha(0 \leq \alpha < 1)$ such that $zf' \in S^*(\alpha)$.

A function $f \in A$ is said to be in the class of β-uniformly convex functions of order α, denoted by $\beta - UCV(\alpha)$ [8, 9] if

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq \beta \left|\frac{zf''(z)}{f'(z)} - 1\right|, \tag{2}$$

and is said to be in a corresponding subclass of $\beta{-}UCV(\alpha)$ denote by $\beta{-}S_p(\alpha)$ if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \beta \left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{3}$$

where $-1 \leq \alpha \leq 1$ and $z \in U$.

The class of uniformly convex and uniformly starlike functions has been extensively studied by Goodman [2, 3], Ma and Minda [7]. In fact the class of uniformly β-starlike funtions was introduced by Kanas and Wisniowski [5], and for which it can be generalised to $\beta - S_p(\alpha)$, the class of uniformly β-starlike functions of order α.

If f of the form (1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ are two functions in A, then the hadamard product (or convolution) of f and g is denoted by $f * g$ and is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{4}$$

Ruscheweyh [10] using the convolution techniques, introduced and studied an important subclass of A, the class of prestarlike functions of order α, which denoted by $\mathcal{R}(\alpha)$. Thus $f \in A$ is said to be prestarlike function of order α $(0 \leq \alpha < 1)$ if $f * S_\alpha \in S^*(\alpha)$ where

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} c_n(\alpha) z^n$$

and $c_n(\alpha) = \frac{\Pi_{j=2}^{n}(j-2\alpha)}{(n-1)!}$ $\quad (n \in \mathbf{N}\ \{1\} \quad \mathbf{N} := \{1, 2, 3, \ldots\})$. We note that $\mathcal{R}(0) = C(0)$ and $\mathcal{R}(\frac{1}{2}) = S^*(\frac{1}{2})$. Juneja et.al [4] define the family $\mathcal{D}(\Phi, \Psi; \alpha)$ consisting of functions $f \in A$ so that

$$\operatorname{Re}\left(\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)}\right) > \alpha, \quad z \in U,$$

where $\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ analytic in U such that $f(z) * \Psi(z) \neq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$ $(n \geq 2)$.

Darus [1] define the family $\mathcal{D}(\Phi, \Psi; \alpha, \beta)$ consisting of functions $f \in A$ such that

$$\operatorname{Re}\left(\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)}\right) > \beta\left|\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - 1\right| + \alpha, \tag{5}$$

where $0 \leq \alpha < 1$ and $\beta \geq 0$. For suitable choices of Φ and Ψ, the various subclasses of A are obtained. For example $\mathcal{D}(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, 0) = S^*(\alpha)$, $\mathcal{D}(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, 0) = C(\alpha)$, $\mathcal{D}(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; \alpha, 0) = \mathcal{R}(\alpha)$. Moreover, $\mathcal{D}(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, \beta) = \beta - S_p(\alpha)$ and $\mathcal{D}(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, \beta) = \beta - UCV(\alpha)$.

Also denote by T [11] the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z_n \tag{6}$$

Now let us write $\mathcal{D}_T(\Phi, \Psi; \alpha, \beta) = \mathcal{D}(\Phi, \Psi; \alpha, \beta) \cap T$ where T is the class of functions of the form (6) that are analytic and univalent in U.

In this paper, we will study the unified presentation of function $f \in T$ belongs to $\mathcal{U}[\Phi, \Psi; \alpha, \beta, \lambda, n]$ which include growth and distortion theorem, radii of convexity, radii of starlikeness and radii of close-to-convexity.

2. COEFFICIENT INEQUALITY

The following result is needed for the purpose of the study.

LEMMA 1. [1] *A function f defined by (6) is in the class $\mathcal{D}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n]}{1-\alpha} |a_n| \leq 1, \tag{7}$$

where $0 \leq \alpha < 1$, $\beta \geq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$.

Next, by observing that

$$f \in \mathcal{E}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta) \Leftrightarrow zf' \in \mathcal{D}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta), \tag{8}$$

we gain the following Lemma 2.

LEMMA 2. *A function f defined by (6) is in the class $\mathcal{E}_T(\Phi, \Psi; \alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{n[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n]}{1-\alpha} |a_n| \leq 1, \tag{9}$$

where $0 \leq \alpha < 1$, $\beta \geq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$.

In view of Lemma 1 and Lemma 2, we unify the classes $\mathcal{D}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta)$ and $\mathcal{E}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta)$ and so a new class $\mathcal{U}(\Phi, \Psi; \alpha, \beta, \lambda, n)$ is formed. Thus we say that a function f defined by (6) belongs to $\mathcal{U}(\Phi, \Psi; \alpha, \beta, \lambda, n)$ if and only if,

$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n]|a_n| \leq 1-\alpha, \tag{10}$$

where $0 \leq \alpha < 1$, $\beta \geq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$, $\lambda \geq 0$ and $\Upsilon_n > \gamma_n$. Clearly, we obtain

$$\mathcal{U}(\Phi, \Psi; \alpha, \beta, \lambda, n) = (1-\lambda)\mathcal{D}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta) + \lambda\mathcal{E}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta),$$

so that

$$\mathcal{U}(\Phi, \Psi; \alpha, \beta, 0, n) = \mathcal{D}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta),$$

and

$$\mathcal{U}(\Phi, \Psi; \alpha, \beta, 1, n) = \mathcal{E}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta).$$

3. GROWTH AND DISTORTION THEOREM

Our first result for function f to be in the class $\mathcal{U}(\Phi, \Psi; \alpha, \beta, \lambda, n)$ is given as follows:

Theorem 1. *Let the function f defined by the formula* (6) *be in the class $\mathcal{U}(\Phi, \Psi; \alpha, \beta, \lambda, n)$, then*

$$|z| - |z|^2 \frac{1-\alpha}{(1+\lambda)[(1+\beta)\Upsilon_2 - (\alpha+\beta)\gamma_2]} \le |f(z)| \le |z| + |z|^2 \frac{1-\alpha}{(1+\lambda)[(1+\beta)\Upsilon_2 - (\alpha+\beta)\gamma_2]} \tag{11}$$

and

$$1 - |z| \frac{2(1-\alpha)}{(1+\lambda)[(1+\beta)\Upsilon_2 - (\alpha+\beta)\gamma_2]} \le |f'(z)| \le 1 + |z| \frac{2(1-\alpha)}{(1+\lambda)[(1+\beta)\Upsilon_2 - (\alpha+\beta)\gamma_2]} \tag{12}$$

The bounds (11) *and* (12) *are attained for functions given by*

$$f(z) = z - z^2 \frac{1-\alpha}{(1+\lambda)[(1+\beta)\Upsilon_2 - (\alpha+\beta)\gamma_2]}. \tag{13}$$

Proof. We find from (10) that

$$\sum_{n=2}^{\infty} |a_n| \le \frac{(1-\alpha)}{(1+\lambda)[(1+\beta)\Upsilon_2 - (\alpha+\beta)\gamma_2]}. \tag{14}$$

Using (6) and (14), we readily have ($z \in U$)

$$\begin{aligned} |f(z)| &\ge |z| - \sum_{n=2}^{\infty} |a_n||z^n| \\ &\ge |z| - |z^2| \sum_{n=2}^{\infty} |a_n| \\ &\ge r - \frac{1-\alpha}{(1+\lambda)[(1+\beta)\Upsilon_2 - (\alpha+\beta)\gamma_2]} r^2, \qquad |z| = r < 1 \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n||z^n| \\ &\leq |z| + |z^2| \sum_{n=2}^{\infty} |a_n| \\ &\leq r + \frac{1-\alpha}{(1+\lambda)[(1+\beta)\Upsilon_2 - (\alpha+\beta)\gamma_2]} r^2, \qquad |z| = r < 1, \end{aligned}$$

which proves the assertion (11) of Theorem 1. Also, from (6), we find for $z \in U$ that

$$\begin{aligned} |f'(z)| &\geq 1 - \sum_{n=2}^{\infty} n|a_n||z^{n-1}| \\ &\geq 1 - |z| \sum_{n=2}^{\infty} n|a_n| \\ &\geq 1 - \frac{2(1-\alpha)}{(1+\lambda)[(1+\beta)\Upsilon_2 - (\alpha+\beta)\gamma_2]} r, \qquad |z| = r < 1 \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\leq 1 + \sum_{n=2}^{\infty} n|a_n||z^{n-1}| \\ &\leq 1 + |z| \sum_{n=2}^{\infty} n|a_n| \\ &\leq 1 + \frac{2(1-\alpha)}{(1+\lambda)[(1+\beta)\Upsilon_2 - (\alpha+\beta)\gamma_2]} r, \qquad |z| = r < 1, \end{aligned}$$

which proves the assertion (12) of Theorem 1. □

4. RADII CONVEXITY AND STARLIKENESS

The radii of convexity for class $\mathcal{U}(\Phi, \Psi; \alpha, \beta, \lambda, n)$ is given by the following theorem.

THEOREM 2. *Let the function f be in the class $\mathcal{U}(\Phi, \Psi; \alpha, \beta, \lambda, n)$. Then the function f is convex of order ρ $(0 \leq \rho < 1)$ in the disk*

$$|z| < r_1(\Phi, \Psi; \alpha, \beta, \lambda, n, \rho) = r_1,$$

where

$$r_1 = \inf_n \left\{ \frac{(1-\rho)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n](1-\lambda+n\lambda)}{n(n-\rho)(1-\alpha)} \right\}^{\frac{1}{n-1}}. \tag{15}$$

Proof. It sufficient to show that

$$\begin{aligned} \left|\frac{zf''(z)}{f'(z)}\right| &= \left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right| \\ &\leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{1-n}} \end{aligned} \tag{16}$$

which implies that

$$\begin{aligned} (1-\rho) - \left|\frac{zf''(z)}{f'(z)}\right| &\geq (1-\rho) - \frac{\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}} \\ &= \frac{(1-\rho)-\sum_{n=2}^{\infty} n(n-\rho)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n-1}}. \end{aligned} \tag{17}$$

Hence, from (15), if

$$|z|^{n-1} \leq \frac{(1-\rho)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n](1-\lambda+n\lambda)}{n(n-\rho)(1-\alpha)}, \tag{18}$$

and according to (10)

$$1-\rho-\sum_{n=2}^{\infty} n(n-\rho)a_n|z|^{n-1} > 1-\rho-(1-\rho) = \rho. \tag{19}$$

Hence, from (19), we obtain

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1-\rho.$$

Therefore

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \rho,$$

which shows that f is convex in the disk $|z| < r_1(\Phi,\Psi;\alpha,\beta,\lambda,n,\rho)$. □

By setting $\lambda = 0$ and $\lambda = 1$, we have the Corollary 1 and the Corollary 2, respectively.

COROLLARY 1. *Let the function f be in the class $\mathcal{D}_{\mathcal{T}}(\Phi,\Psi;\alpha,\beta)$. Then the function f is convex of order ρ $(0 \leq \rho < 1)$ in the disk*

$$|z| < r_2(\Phi,\Psi;\alpha,\beta,0,n,\rho) = r_2,$$

where

$$r_2 = \inf_n \left\{ \frac{(1-\rho)[(1+\beta)\Upsilon_2 - (\alpha+\beta)\gamma_2]}{n(n-\rho)(1-\alpha)} \right\}^{\frac{1}{n-1}}. \tag{20}$$

COROLLARY 2. *Let the function f be in the class $\mathcal{E}_{\mathcal{T}}(\Phi,\Psi;\alpha,\beta)$. Then the function f is convex of order ρ $(0 \le \rho < 1)$ in the disk*

$$|z| < r_3(\Phi,\Psi;\alpha,\beta,1,n,\rho) = r_3,$$

where

$$r_3 = \inf_n \left\{ \frac{(1-\rho)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n]}{(n-\rho)(1-\alpha)} \right\}^{\frac{1}{n-1}}. \tag{21}$$

THEOREM 3. *Let the function f be in the class $\mathcal{U}(\Phi,\Psi;\alpha,\beta,\lambda,n)$. Then the function f is starlike of order ρ $(0 \le \rho < 1)$ in the disk*

$$|z| < r_4(\Phi,\Psi;\alpha,\beta,\lambda,n,\rho) = r_4,$$

where

$$r_4 = \inf_n \left\{ \frac{(1-\rho)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n](1-\lambda+n\lambda)}{(n-\rho)(1-\alpha)} \right\}^{\frac{1}{n-1}}. \tag{22}$$

Proof. It sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1-\rho$$

Using similar method of Theorem 2 and making use of (10), we get (22). □

Letting $\lambda = 0$ and $\lambda = 1$, we have the Corollary 3 and the Corollary 4, respectively.

COROLLARY 3. *Let the function f be in the class $\mathcal{D}_{\mathcal{T}}(\Phi,\Psi;\alpha,\beta,n)$. Then the function f is starlike of order ρ $(0 \le \rho < 1)$ in the disk*

$$|z| < r_5(\Phi,\Psi;\alpha,\beta,0,n,\rho) = r_5,$$

where

$$r_5 = \inf_n \left\{ \frac{(1-\rho)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n]}{(n-\rho)(1-\alpha)} \right\}^{\frac{1}{n-1}}. \tag{23}$$

COROLLARY 4. *Let the function f be in the class $\mathcal{E}_{\mathcal{T}}(\Phi,\Psi;\alpha,\beta,n)$. Then the function f is starlike of order ρ $(0 \le \rho < 1)$ in the disk*

$$|z| < r_6(\Phi,\Psi;\alpha,\beta,1,n,\rho) = r_6,$$

where

$$r_6 = \inf_n \left\{ \frac{n(1-\rho)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n]}{(n-\rho)(1-\alpha)} \right\}^{\frac{1}{n-1}}. \tag{24}$$

Last, but not least we give the following result.

THEOREM 4. *Let the function f be in the class $\mathcal{U}(\Phi,\Psi;\alpha,\beta,\lambda,n)$. Then the function f is close-to-convex of order $\rho (0 \le \rho < 1)$ in the disk*

$$|z| < r_7(\Phi,\Psi;\alpha,\beta,\lambda,n,\rho) = r_7,$$

where

$$r_7 = \inf_n \left\{ \frac{(1-\rho)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n](1-\lambda+n\lambda)}{n(1-\alpha)} \right\}^{\frac{1}{n-1}}. \tag{25}$$

Proof. It sufficient to show that

$$|f'(z) - 1| < 1 - \rho.$$

Using similar technique of Theorem 2 and making use of (10), we get (25). □

5. INTEGRAL TRANSFORM OF THE CLASS $\mathcal{U}(\Phi,\Psi;\alpha,\beta,\lambda,N)$

For $f \in A$ we define the integral transform

$$V_\mu(f)(z) = \int_0^1 \mu(t) \frac{f(tz)}{t} \mathrm{d}t,$$

where μ is real valued, non-negative weight function normalized such that $\int_0^1 \mu(t)\mathrm{d}t = 1$. Since special cases of $\mu(t)$ are particularly interesting such as $\mu(t) = (1+c)t^c$, $c > -1$, for which V_μ is known as the Bernardi operator, and

$$\mu(t) = \frac{(c+1)^\delta}{\mu(\delta)} t^c \left(\log \frac{1}{t}\right)^{\delta-1}, \; c > -1, \; \delta \ge 0$$

which gives the Komatu operator. For more details see [6].

First of all, we show that the class $\mathcal{U}(\Phi,\Psi;\alpha,\beta,\lambda,n)$ is closed under $V_\mu(f)$.

THEOREM 5. *Let $f \in \mathcal{U}(\Phi,\Psi;\alpha,\beta,\lambda,n)$. Then $V_\mu(f) \in \mathcal{U}(\Phi,\Psi;\alpha,\beta,\lambda,n)$.*

Proof. By definition, we can write

$$\begin{aligned} V_\mu(f) &= \frac{(c+1)^\delta}{\mu(\delta)} \int_0^1 (-1)^{\delta-1} t^c (\log t)^{\delta-1} \left(z - \sum_{n=2}^\infty a_n z^n t^{n-1} \right) \mathrm{d}t \\ &= \frac{(-1)^{\delta-1}(c+1)^\delta}{\mu(\delta)} \lim_{r\to 0^+} \left[\int_r^1 t^c (\log t)^{\delta-1} \left(z - \sum_{n=2}^\infty a_n z^n t^{n-1} \right) \mathrm{d}t \right], \end{aligned}$$

and a simple calculation gives

$$V_\mu(f)(z) = z - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^{\delta} a_n z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty} \frac{(1-\lambda+n\lambda)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n]}{1-\alpha} \left(\frac{c+1}{c+n}\right)^{\delta} a_n < 1. \tag{26}$$

On the other hand by (10), $f \in \mathcal{U}(\Phi, \Psi; \alpha, \beta, \lambda, n)$ if and only if

$$\sum_{n=2}^{\infty} \frac{(1-\lambda+n\lambda)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n]}{1-\alpha} < 1. \tag{27}$$

Hence $\frac{c+1}{c+n} < 1$. Therefore (26) holds true and the proof is complete. □

Next we provide a starlikeness condition for functions in $\mathcal{U}(\Phi, \Psi; \alpha, \beta, \lambda, n)$ under $V_\mu(f)$.

THEOREM 6. *Let $f \in \mathcal{U}(\Phi, \Psi; \alpha, \beta, \lambda, n)$. Then $V_\mu(f)$ is starlike of order $0 \le \rho < 1$ in $|z| < R_8$ where*

$$R_8 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\rho)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n](1-\lambda+n\lambda)}{(n-\rho)(1-\alpha)}\right]^{\frac{1}{n-1}}$$

Proof. It is sufficient to prove

$$\left|\frac{z(V_\mu(f)(z))'}{V_\mu(f)(z)} - 1\right| < 1-\rho. \tag{28}$$

For the left hand side of (28) we have

$$\begin{aligned} \left|\frac{z(V_\mu(f)(z))'}{V_\mu(f)(z)} - 1\right| &= \left|\frac{\sum_{n=2}^{\infty}(1-n)(\frac{c+1}{c+n})^{\delta} a_n z^{n-1}}{1-\sum_{n=2}^{\infty}(\frac{c+1}{c+n})^{\delta} a_n z^{n-1}}\right| \\ &\le \frac{\sum_{n=2}^{\infty}(n-1)(\frac{c+1}{c+n})^{\delta} a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty}(\frac{c+1}{c+n})^{\delta} a_n |z|^{n-1}}. \end{aligned}$$

This last expression is less than $(1-\rho)$ since

$$|z|^{n-1} < \left(\frac{c+1}{c+n}\right)^{\delta} \frac{(1-\rho)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n](1-\lambda+n\lambda)}{(n-\rho)(1-\alpha)}.$$

Therefore the proof is complete. □

Using the fact that f is convex if and only if zf' is starlike, we obtain the following:

Theorem 7. *Let $f \in \mathcal{U}(\Phi, \Psi; \alpha, \beta, \lambda, n)$. Then $V_\mu(f)$ is convex of order $0 \le \rho < 1$ in $|z| < R_9$, where*

$$R_9 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\rho)[(1+\beta)\Upsilon_n - (\alpha+\beta)\gamma_n](1-\lambda+n\lambda)}{n(n-\rho)(1-\alpha)} \right]^{\frac{1}{n-1}}.$$

We omit the proof as it is easily derived.

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