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BESSEL TRANSFORMS AND HARDY SPACE OF GENERALIZED BESSEL FUNCTIONS

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Abstract. In this paper, which was motivated by the papers of S. Ponnusamy [11, 12], we continue the study of generalized and normalized Bessel functions of the first kind of real order. We present some immediate applications of convexity and univalence involving Bessel functions associated with the Hardy space of analytic functions, i.e. we obtain conditions for the function

$$u_p(z) = \sum_{n=0}^{\infty} \left(-\frac{c}{4}\right)^n \frac{\Gamma\left(p+\frac{b+1}{2}\right)}{\Gamma\left(p+n+\frac{b+1}{2}\right)} \frac{z^n}{n!},\ b, p, c \in \mathbb{R}, z \in \mathbb{C}$$

to belong to the Hardy space $\mathcal{H}^\infty$. Let consider $\mathcal{A}$, the class of all analytic and normalized functions in the unit disk and

$$\mathcal{R}(\alpha) = \{f \in \mathcal{A} : \exists\ \eta \in \mathbb{R} \text{ such that } \operatorname{Re}\,[\mathrm{e}^{\mathrm{i}\eta}(f'(z) - \alpha)] > 0, z \in U\}.$$

When $\eta = 0$ we denote $\mathcal{R}(\alpha)$ simply by $\mathcal{R}_0(\alpha)$, and when $\alpha = 0$, we denote $\mathcal{R}_0(\alpha)$ simply by $\mathcal{R}$. We find conditions for the convolution $zu_p(z) * f(z)$ to belong to $\mathcal{H}^\infty \cap \mathcal{R}$, where f is an analytic function in $\mathcal{R}$. Finally we obtain conditions for α_1, α_2 and the parameters b, c, p such that the operator $B(f) := zu_p(z) * f(z)$ maps $\mathcal{R}(\alpha_1)$ into $\mathcal{R}(\alpha_2)$.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be the set of all analytic functions on the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. $\mathcal{H}^\infty$ denotes the space of all bounded functions on $\mathcal{H}$. This is a Banach algebra w.r.t. the norm $||f||_\infty = \sup\{|f(z)| : z \in U\}$. For $p \in (0, \infty)$, we denote by $\mathcal{H}^p$ the space of all functions $f \in \mathcal{H}$ such that $|f|^p$ admits a harmonic majorant. $\mathcal{H}^p$ is a Banach space if we define the norm of f to be the p-th root of the least harmonic majorant of $|f|^p$, taken in some fixed point $z \in U$. For the function $f \in \mathcal{H}$, set

$$M_p(r, f) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta\right)^{1/p},\ p \in (0, \infty) \\ \max\{|f(z)| : |z| \le r\},\ p = \infty \end{cases} \tag{1.1}$$

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The function f is said to belong to $\mathcal{H}^p$ (which is called the Hardy space) if $M_p(r,f)$ is bounded for all $r\in[0,1)$. Two widely known results [3] about the Hardy spaces $\mathcal{H}^p$ are the followings (the first is due to Smirnow [13] and the second is due to Hardy and Littlewood [7]):

$$\operatorname{Re} f'(z)>0 \Rightarrow \begin{cases} f'\in\mathcal{H}^q,\ \forall q<1 \\ f\in\mathcal{H}^{q/(1-q)},\ \forall q\in(0,1). \end{cases} \tag{1.2}$$

Let $\mathcal{A}$ denote the class of functions of the form $f(z)=z+\sum_{n\geq 2}a_nz^n$, which are analytic in U. We denote by $\mathcal{C}(\alpha),\mathcal{S}^*(\alpha)$ the well known subsets of $\mathcal{A}$, which are convex of order α, respectively starlike of order α. When $\alpha=0$ we denote $\mathcal{C}(\alpha)$ and $\mathcal{S}^*(\alpha)$ simply by $\mathcal{C}$ and $\mathcal{S}^*$, respectively. A function f, analytic in the unit disk U, is said to be convex if it is univalent and $f(U)$ is a convex domain. It is well known that f is convex [4] if and only if $f'(0)\neq 0$ and

$$\operatorname{Re}\left[1+zf''(z)/f'(z)\right]>0,\ z\in U. \tag{1.3}$$

If in addition

$$\operatorname{Re}\left[1+zf''(z)/f'(z)\right]>\alpha,\ z\in U, \tag{1.4}$$

where $\alpha\in[0,1)$, then f is called convex of order α. The function g with $g(0)=0$ and $g'(0)\neq 0$ is starlike [4] if and only if

$$\operatorname{Re}\left[zg'(z)/g(z)\right]>0,\ z\in U. \tag{1.5}$$

If in addition

$$\operatorname{Re}\left[zg'(z)/g(z)\right]>\alpha,\ z\in U, \tag{1.6}$$

where $\alpha\in[0,1)$, then g is called *starlike of order* α. For $\alpha<1$ we also introduce the class

$$\mathcal{P}(\alpha)=\{p\in\mathcal{H}:\exists\ \eta\in\mathbb{R}\ \text{s.t.}\ p(0)=1,\operatorname{Re}\ [\mathrm{e}^{\mathrm{i}\eta}(p(z)-\alpha)]>0,z\in U\} \tag{1.7}$$

and define $\mathcal{R}(\alpha)=\{f\in\mathcal{A}:f'(z)\in\mathcal{P}(\alpha)\}$, i.e.

$$\mathcal{R}(\alpha)=\{f\in\mathcal{A}:\exists\ \eta\in\mathbb{R}\ \text{such that}\ \operatorname{Re}\ [\mathrm{e}^{\mathrm{i}\eta}(f'(z)-\alpha)]>0,z\in U\}. \tag{1.8}$$

When $\eta=0$ we denote $\mathcal{P}(\alpha)$ and $\mathcal{R}(\alpha)$ simply by $\mathcal{P}_0(\alpha)$ and $\mathcal{R}_0(\alpha)$, respectively; for $\alpha=0$ we denote $\mathcal{P}_0(\alpha)$ and $\mathcal{R}_0(\alpha)$ simply by $\mathcal{P}$ and $\mathcal{R}$, respectively. Finally, the convolution or Hadamard product of two power series $f(z)=z+\sum_{n\geq 2}a_nz^n$, and $g(z)=z+\sum_{n\geq 2}b_nz^n$, is the power series $(f*g)(z)=z+\sum_{n\geq 2}a_nb_nz^n$. We continue these preliminaries with the next condition of univalence (close-to-convexity [4]) due to Ozaki [9] and Kaplan [8].

Lemma 1.9. [8], [9] *Let D be a simply connected domain and f an analytic function in D. If there exists a function φ, univalent in D such that $\varphi(D)$ is a convex domain and* $\operatorname{Re}[f'(z)/\varphi'(z)]>0$, *for all $z\in D$, i.e. f is close-to-convex, then f is univalent in D.*

The next lemmas will be used to prove several theorems.

LEMMA 1.10. [6] *Let consider the analytic function* $g(z) = \sum_{n\geq 1} A_n z^{n-1}$, *where* $A_1 = 1$ *and* $A_n \geq 0$, *for all* $n \geq 2$. *If* $\{A_n\}$ *is a convex decreasing sequence, i.e.,* $A_n - 2A_{n+1} + A_{n+2} \geq 0$ *and* $A_n - A_{n+1} \geq 0$, *for all* $n \geq 1$, *then* $\mathrm{Re}[g(z)] > 1/2$, *for all* $z \in U$.

LEMMA 1.11. [14] $\mathcal{P}_0(\alpha) * \mathcal{P}_0(\beta) \subset \mathcal{P}_0(\gamma)$, *where* $\gamma = 1 - 2(1-\alpha)(1-\beta)$ *and* $\alpha, \beta < 1$. *The value of* γ *is the best possible.*

LEMMA 1.12. [10] *For* $\alpha, \beta < 1$ *and* $1 - \gamma = 2(1-\alpha)(1-\beta)$, *we have* $\mathcal{R}(\alpha) * \mathcal{R}_0(\beta) \subset \mathcal{R}(\gamma)$, *or equivalently* $\mathcal{P}(\alpha) * \mathcal{P}_0(\beta) \subset \mathcal{P}(\gamma)$.

LEMMA 1.13. [5] *If the function* f, *convex of order* α, *where* $\alpha \in [0, 1)$, *is not of the form*

$$\left\{ \begin{array}{ll} f(z) = a + bz(1 - z\mathrm{e}^{\mathrm{i}\gamma})^{2\alpha - 1}, & \alpha \neq 1/2 \\ f(z) = a + b\log(1 - z\mathrm{e}^{\mathrm{i}\gamma}), & \alpha = 1/2 \end{array} \right. \tag{1.14}$$

for some complex numbers a *and* b, *and for some real number* γ, *then the following statements hold:*

(i) *There exists* $\delta = \delta(f) > 0$ *s.t.* $f' \in \mathcal{H}^{\delta + 1/[2(1-a)]}$;
(ii) *If* $\alpha \in [0, 1/2)$, *then there exists* $\tau = \tau(f) > 0$ *s.t.* $f \in \mathcal{H}^{\tau + 1/(1-2\alpha)}$;
(iii) *If* $\alpha \geq 1/2$, *then* $f \in \mathcal{H}^{\infty}$.

We recall that the generalized Bessel function w_p (cf. [1, 2]) is defined as a particular solution of the linear differential equation

$$z^2 w''(z) + bzw'(z) + \left[cz^2 - p^2 + (1-b)p\right] w(z) = 0, \tag{1.15}$$

where $b, p, c \in \mathbb{R}$. The analytic function w_p has the form

$$w_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!\Gamma\left(p + n + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p}, z \in \mathbb{C}. \tag{1.16}$$

Now, the generalized and normalized Bessel function u_p is defined with the transformation $u_p(z) = [a_0(p)]^{-1} z^{-p/2} w_p(z^{1/2})$, where

$$a_0(p) = \left[2^p \Gamma\left(p + \frac{b+1}{2}\right)\right]^{-1}.$$

Using the Pochhammer symbol, defined in terms of Γ-functions by $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\ldots(a+n-1)$ and $(a)_0 = 1$, we obtain for the function u_p the following form

$$u_p(z) = \sum_{n\geq 0} \frac{(-1)^n c^n}{4^n (\kappa)_n} \frac{z^n}{n!}, \tag{1.17}$$

where $\kappa = p + (b+1)/2 \neq 0, -1, -2, \ldots$. For this function we have the following result, which is the first part of Theorem 3.3, [2]:

LEMMA 1.18. *Let be* $\alpha \in [0,1), b, p, c \in \mathbb{R}$ *such that* $\lambda = (1-c^2)\alpha^2 - 2\alpha + 1 > 0$ *and* $\kappa \geq (1-\alpha)|c|/4\sqrt{\lambda} + 1$. *Therefore we have* $\mathrm{Re}[u_p(z)] > \alpha$, $\forall\ z \in U$.

We end this section with the following lemma. The first part of this lemma is actually a part of Theorem 4.9, [2], and the second part is Corollary 3.6, [2].

LEMMA 1.19. *Let* $b, p \in \mathbb{R}$ *such that* $\kappa = p + (b+1)/2 \neq 0$. *The following implications are true in the unit disk:*

(i) *If for* $c < 0$, $N(c) = [-(c+2) + \sqrt{c^2/2 - 4c + 4}]/2$ *we have* $\kappa \geq N(c)/2$, *then* $\mathrm{Re}[u_p(z)] > 1/2$.

(ii) *If for* $|c| < 1$ *we have* $\kappa \geq |c|/\sqrt{1-c^2} + 1$, *then* $\mathrm{Re}[u_p(z)] > 1/2$.

2. HARDY SPACE OF BESSEL FUNCTIONS

In this section we determine conditions for the Bessel function u_p to belong to the space $\mathcal{H}^\infty$, moreover we find conditions for the convolution $zu_p(z) * f(z)$ to belong to $\mathcal{H}^\infty \cap \mathcal{R}$, where f is an analytic function in $\mathcal{R}$.

THEOREM 2.1. *Let* $\alpha \in [0,1)$, $b, p, c \in \mathbb{R}$ *s.t.* $4\alpha^2 + (|c| - 8)\alpha + 2 \geq 0$ *and* $\kappa = p + (b+1)/2 \neq 0$. *If* $\kappa \geq [|c| + 2(2\alpha^2 - 4\alpha + 1)]/[4(1-\alpha)]$, *then*

(i) *for* $\alpha \geq 1/2$ *we have* $u_p \in \mathcal{H}^\infty$;

(ii) *for* $\alpha \in [0, 1/2)$ *we have* $u_p \in \mathcal{H}^{1/(1-2\alpha)}$.

Proof. First we observe that

$$\frac{1}{(1 - z\mathrm{e}^{\mathrm{i}\gamma})^{1-2\alpha}} = {}_2F_1(1, 1-2\alpha, 1; z\mathrm{e}^{\mathrm{i}\gamma}) \tag{2.2}$$

for $\alpha \neq 1/2$ and for real γ, where by the definition of the Gaussian hypergeometric function, the function ${}_2F_1(a,b,c;z)$ is the following hypergeometric series

$$ {}_2F_1(a,b,c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \cdots + \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} + \ldots\ . \tag{2.3}$$

Therefore the function u_p is not of the forms $(1 - z\mathrm{e}^{\mathrm{i}\gamma})^{2\alpha-1}$ (for $\alpha \neq 1/2$) and $\log(1 - z\mathrm{e}^{\mathrm{i}\gamma})$ (for $\alpha = 1/2$). We know that by assumption the normalized Bessel function is convex of order α (first part of Theorem 3.8, [2]). Hence by Lemma 1.13 and part (i) of Theorem 3.8, [2], the proof is completed. $\square$

THEOREM 2.4. *If* $\kappa \geq |c|/4 - 1/2$, *where* $b, p, c \in \mathbb{R}$, *then* $u_p \in \mathcal{H}^\infty$.

Proof. From Proposition 2.17, [2] we have that

$$4\kappa u_p'(z) = -cu_{p+1}(z). \tag{2.5}$$

Also by the hypergeometric representation we observe that $u_{p+1}(z)$ is not of the form $(1 - z\mathrm{e}^{\mathrm{i}\gamma})^{-1}$. Combining (2.5) and part (iii) of Theorem 3.1, [2] (or part (i) of Theorem 3.8 for $\alpha = 0$, [2]), we obtain that u_p' is convex in the unit disk and is not of the form $(1 - z\mathrm{e}^{\mathrm{i}\gamma})^{-1}$. Hence by Lemma 1.13 we have

$u_p' \in \mathcal{H}^1$. Therefore using Theorem 3.11, [3], we deduce that u_p is continuous in $\overline{U} = U \cup \partial U = \{z \in \mathbb{C} : |z| \le 1\}$, so u_p is a bounded analytic function in U. This completes the proof of Theorem 2.4. $\square$

Theorem 2.6. *Let $b, p \in \mathbb{R}$ such that $\kappa = p + (b+1)/2 \neq 0$. For*

(i) $c < 0$, *let* $\kappa \ge N(c)/2$, *where* $N(c) = [-(c+2) + \sqrt{c^2/2 - 4c + 4}]/2$;

(ii) $|c| < 1$, *let be* $\kappa \ge |c|/\sqrt{1-c^2} + 1$.

If $f \in \mathcal{R}$, then the convolution $zu_p(z) * f(z)$ is in $\mathcal{H}^\infty \cap \mathcal{R}$.

Proof. Define $g(z) = zu_p(z) * f(z)$. Then $g'(z) = u_p(z) * f'(z)$. By the hypotheses and Lemma 1.19 we have $\operatorname{Re} u_p(z) > 1/2$. Since $f \in \mathcal{R}$, it follows from Lemma 1.11 that $g \in \mathcal{R}$. Now it is clear that the function u_p is an entire function and consequently the function g is itself an entire function. Consequently the function g is bounded, which completes the proof. $\square$

Remark 2.7. In fact there is an other argument for the proof of Theorem 2.6. From the previous proof it is clear that $g \in \mathcal{R}$. Thus we have by the first implication of (1.2) $g' \in \mathcal{H}^q$ for all $q < 1$. Hence by a well known result of Hardy and Littlewood, i.e., the second implication of (1.2), we have $g \in \mathcal{H}^{q/(1-q)}$ for all $q \in (0,1)$, or equivalently, $g \in \mathcal{H}^p$ for all $p \in (0,\infty)$.

Using the well known bound for Carathéodory functions, we obtain that if $f(z) = z + \sum_{n\ge 2} a_n z^n \in \mathcal{R}$, then $n|a_n| \le 2$, for all $n \ge 2$. Using this fact we find that

$$\begin{aligned}
|g(z)| =& \left| z + \sum_{n\ge 2} b_{n-1} a_n z^n \right| = \left| z + \sum_{n\ge 2} \left(-\frac{c}{4}\right)^{n-1} \frac{a_n}{(\kappa)_{n-1}} \frac{z^n}{(n-1)!} \right| \\
\le& |z| + \sum_{n\ge 2} \left| \left(-\frac{c}{4}\right)^{n-1} \frac{1}{(\kappa)_{n-1}(n-1)!} \right| |a_n||z|^n \\
<& 1 + \sum_{n\ge 2} \left| -\frac{c}{4} \right|^{n-1} \frac{1}{(\kappa)_{n-1}(n-1)!} \frac{2}{n} \\
=& 1 + 2\sum_{n\ge 1} \left| -\frac{c}{4} \right|^{n} \frac{1}{(\kappa)_{n}(n)!} \frac{1}{n+1} \\
<& 1 + 2\sum_{n\ge 1} \left| -\frac{c}{4} \right|^{n} \frac{1}{(\kappa)_{n}(n)!} \frac{1}{n}.
\end{aligned}$$

Therefore, applying d'Alembert's test for convergence, we deduce that the above series

$$\sum_{n\ge 1} \left| -\frac{c}{4} \right|^{n} \frac{1}{(\kappa)_{n}(n)!} \frac{1}{n} 1^n \tag{2.8}$$

converges absolutely for $|z| = 1$. This argument shows that the power series for $g(z)$ converges absolutely. Further, it is known (by Theorem 3.11, [3]) that $g' \in \mathcal{H}^q$ implies the continuity of g on $\overline{U}$, the closure of U. Finally, since the continuous function g on the compact set $\overline{U}$ is bounded, g is a bounded analytic function in U. Therefore, $g \in \mathcal{H}^\infty$, and this completes the proof.

3. BESSEL TRANSFORMS OF FUNCTIONS WITH DERIVATIVE IN A HALF PLANE

The main aim of this section is to find conditions on α_1 and α_2 and the parameters b, c and p such that the operator $B(f) = zu_p(z) * f(z)$ maps $\mathcal{R}(\alpha_1)$ into $\mathcal{R}(\alpha_2)$.

THEOREM 3.1. *Let* $\alpha \in [0, 1)$, $b, p, c \in \mathbb{R}$ *s.t.* $\lambda = (1 - c^2)\alpha^2 - 2\alpha + 1 > 0$ *and* $\kappa \geq (1 - \alpha)|c|/\sqrt{\lambda} + 1$. *If we have* $f \in \mathcal{R}(\alpha_1)$, *where* $\alpha_1 < 1$, *then the Bessel transform* $zu_p(z) * f(z) \in \mathcal{R}(\gamma)$, *where* $\gamma = 1 - 2(1 - \alpha_1)(1 - \alpha)$.

Proof. Let $f \in \mathcal{R}(\alpha_1)$. Define the convolution (or Bessel transform) of the function f by $g(z) = zu_p(z) * f(z)$. Then we have $g'(z) = u_p(z) * f'(z)$. By the hypotheses and Lemma 1.18, we have that $u_p \in \mathcal{P}_0(\alpha)$. Using part (i) of Lemma 1.12 and the fact that $f' \in \mathcal{P}(\alpha_1)$, we immediately get that the function g' belongs to $\mathcal{P}(\gamma)$, where $\gamma = 1 - 2(1 - \alpha_1)(1 - \alpha)$. But the fact that $g' \in \mathcal{P}(\gamma)$ is equivalent to $g \in \mathcal{R}(\gamma)$, therefore the proof is complete. $\square$

As an immediate consequence of Theorem 3.1 we have for $\gamma = 0$.

COROLLARY 3.2. *Under the hypotheses of Theorem* 3.1 *with* $\gamma = 0$ *we have that if* $f \in \mathcal{R}(\alpha_1)$, *then* $zu_p(z) * f(z) \in \mathcal{R}(0)$, *where* $\alpha_1 = (1 - 2\alpha)/(2 - 2\alpha)$.

Taking $\alpha = 0$ in the above Corollary we have the next result.

COROLLARY 3.3. *Let* $b, p \in \mathbb{R}$ *such that* $\kappa \geq |c|/4+1$. *If we have* $f \in \mathcal{R}(1/2)$, *then* $zu_p(z) * f(z) \in \mathcal{R}(0)$.

In the following we present the analogous of Theorem 3.1, using Lemma 1.10 of Fejér. First we find conditions for p, b, c, α such that $\mathrm{Re}[u_p(z)] > \alpha$ and with a similar procedure we obtain conditions for the parameters such that the operator $B(f)$ maps $\mathcal{R}(\alpha_1)$ into $\mathcal{R}(\alpha_2)$. We prove first the next lemma.

LEMMA 3.4. *Let* $b, p \in \mathbb{R}$, $c < 0$ *and* $\alpha \in [0, 1)$. *Let* $c_0 = c/[8(\alpha - 1)]$, $c_1 = [-13 + \sqrt{77 - 2c}]/2$ *and* $c_2 = [-(c/2 + 27) + \sqrt{c^2/4 - 23c + 169}]/10$. *Suppose that* $\kappa \geq \max\{c_0, c_1, c_2\}$, *where* $\kappa = p + (b + 1)/2$. *Further, let* b_n *be the* n*-th coefficient of the generalized and normalized Bessel function, i.e.* $b_n = (-c/4)^n/[(\kappa)_n n!]$ *for all* $n \in \mathbb{N}$. *Then the sequence* $\{A_n\}$, *defined by* $A_n = b_{n-1}/[2(1 - \alpha)]$ *for all* $n \geq 2$ *and* $A_1 = 1$, *is a nonnegative convex decreasing sequence.*

Proof. First we prove that the sequence $\{A_n\}$ is decreasing. The inequality $A_2 \leq A_1$ is equivalent to $\kappa \geq c_0 = c/[8(\alpha - 1)]$. Now we want to prove that $A_n - A_{n+1} \geq 0$ for all $n \geq 2$. By definition $A_n - A_{n+1} = [b_{n-1} - b_n]/[2(1 - \alpha)]$,

$n \geq 2$, therefore using the recursive relation $4n(\kappa+n-1)b_n = -cb_{n-1}$, we obtain that

$$A_n - A_{n+1} = \frac{[4n^2 + 4(\kappa-1)n + c]b_{n-1}}{8(1-\alpha)n(\kappa+n-1)},\ n \geq 2. \tag{3.5}$$

Let denote $M_1(n) = 4n^2 + 4(\kappa-1)n + c$. Because $(n-2)^2 \geq 0$, we have that $n^2 \geq 4n - 4$, for all $n \geq 2$. This means that $M_1(n) \geq 4(\kappa+3)n + c - 16$. By hypothesis $\kappa > 0$, therefore $M_1(n) \geq 4(\kappa+3)n + c - 16 \geq M_1(2) = 8\kappa + c + 8$ and this value is positive, because $8\kappa \geq c/(\alpha-1) \geq -c-8$. Clearly, by the assumptions we get that the sequence is nonnegative. Therefore, we need only to show that the hypotheses imply that

$$A_n - 2A_{n+1} + A_{n+2} \geq 0, \quad \text{for all} \quad n \geq 1. \tag{3.6}$$

We observe that the condition $A_1 - 2A_2 + A_3 \geq 0$ is equivalent to the inequality $16\kappa + c + 16 \geq 0$. But $16\kappa + c + 16 \geq 8(\kappa+1) > 0$ and therefore, by the assumptions, the inequality (3.6) holds for $n = 1$. Next we verify the inequality (3.6) for $n \geq 2$. From the definition of A_n, we find that

$$A_n - 2A_{n+1} + A_{n+2} = \frac{b_{n-1}M_2(n)}{4(1-\alpha)n(n+1)(\kappa+n-1)(\kappa+n)}, \tag{3.7}$$

where $M_2(n) = 2n^4 + 4\kappa n^3 + D_1 n^2 + D_2 n + D_3$ with $D_1 = 2\kappa^2 + 2\kappa + c - 2$, $D_2 = 2\kappa^2 + (c-2)\kappa + c$ and $D_3 = (8\kappa + c)c/8$. With some computation we find that

$$A_n - 2A_{n+1} + A_{n+2} = \frac{b_{n-1}M_3(n)}{2(1-\alpha)n(n+1)(\kappa+n-1)(\kappa+n)}, \tag{3.8}$$

where $M_3(n) = (n-2)^4 + 2(\kappa+4)(n-2)^3 + E_1(n-2)^2 + E_2(n-2) + E_3$, with the following expressions

$$\begin{cases} E_1 = \kappa^2 + 13\kappa + c/2 + 23 \\ E_2 = 5\kappa^2 + (c/2 + 27)\kappa + 5c/2 + 28 \\ E_3 = 9\kappa^2 + (3c/2 + 57)\kappa + c^2/16 + 9c/2 + 81. \end{cases}$$

Let first $c \in [-37, 0)$ and let $c_1 = [-13 + \sqrt{77 - 2c}]/2$. It is easy to check that the equation $E_1 = 0$ (with variable κ) have the greatest real root c_1, which is negative. Therefore $c_1 < c/[8(\alpha-1)] \leq \kappa$, which means that E_1 is nonnegative. Analogously $c_2 = [-(c/2 + 27) + \sqrt{c^2/4 - 23c + 169}]/10$ is the greatest real root of the equation $E_2 = 0$. Because $c \in [-37, 0)$, c_2 is also negative, therefore $c_2 < \kappa$, consequently E_2 is nonnegative. The equation $E_3 = 0$ has the real roots $[-(c/2+19) \pm \sqrt{c+37}]/6$, and for this we need $c \in [-37, 0)$. The greatest root $c_3 = [-(c/2 + 19) + \sqrt{c+37}]/6$ is nonnegative for $c \in [-37, 0)$. Moreover it is less than $c/[8(\alpha-1)] \leq \kappa$, which means that E_3 is also nonnegative. Now let $c < -37$, then clearly E_3 is nonnegative and because $\kappa \geq \max\{c_0, c_1, c_2\}$, we have that E_1 and E_2 is also positive. Therefore we see that E_i, $i \in \{1, 2, 3\}$ are nonnegative. From this observation we deduce that $M_3(n) \geq 0$, for all $n \geq 2$. Thus $\{A_n\}$ is a convex decreasing sequence. $\square$

Using the above Lemma we obtain the next result.

THEOREM 3.9. *Let* $\alpha \in [0,1)$, $b, p \in \mathbb{R}$, $c < 0$ *such that* $\kappa = p + (b+1)/2 \geq \max\{c_0, c_1, c_2\}$, *(where* c_0, c_1, c_2 *are defined in Lemma 3.4) then* $\mathrm{Re}[u_p(z)] > \alpha$, *i.e.* $u_p \in \mathcal{P}_0(\alpha)$.

Proof. Let A_n be defined as in Lemma 3.4. Then the condition $\mathrm{Re}[u_p(z)] > \alpha$ is equivalent to

$$\mathrm{Re}\left[1 + \sum_{n\geq 2} A_n z^{n-1}\right] > 1/2, \; z \in U. \tag{3.10}$$

By Lemma 3.4 and the hypotheses, we observe that the sequence $\{A_n\}$ is convex decreasing and therefore the conclusion follows if we apply Lemma 1.10. □

We know that if $c \in [-37, 0)$, then $\max\{c_0, c_1, c_2\} = c_0$, therefore taking $\alpha = 0$ in Theorem 3.9 we immediately get that.

THEOREM 3.11. *If* $\kappa \geq -c/8 - 1$ *for* $b, p \in \mathbb{R}$, $c \in [-37, 0)$, *then* u_p *is univalent in the unit disk.*

Proof. If we apply Theorem 3.9 for $\alpha = 0$ and Lemma 1.9 for the special case $D = U$, $\varphi(z) = -(cz)/(4\kappa)$, we obtain the univalence condition for generalized and normalized Bessel functions of the first kind of p order. Therefore, if $\kappa \geq -c/8 - 1$, we obtain that $\mathrm{Re}\, u_{p+1}(z) > 0$, for all $z \in U$. Using Proposition 2.17, [2], we conclude that

$$\mathrm{Re}\, u_{p+1}(z) = \mathrm{Re}\left[-\frac{4\kappa}{c} u_p'(z)\right] > 0, \forall\; z \in U, \tag{3.12}$$

which means that u_p is close-to-convex of order 0, i.e. it is univalent in U. □

REMARK 3.13. In Corollary 4.18, [2], we see that for arbitrary b and $c < 0$, if $\kappa \geq -c/4 - 1$, then u_p is univalent. Clearly we have $-c/4 - 1 > -c/8 - 1$, therefore, for $c \in [-37, 0)$ the result of Theorem 3.11 is better than the result of Corollary 4.18, [2]. Therefore an interesting open question is the following: Find the explicit range of b, c, p for which the function u_p is univalent in U.

Now we have the following consequences of Theorem 3.9.

COROLLARY 3.14. *Let* $\alpha \in [0,1)$, $b \in \mathbb{R}$, $c \in [-37, 0)$ *and suppose that* $8(1-\alpha)(\kappa+1) + c \geq 0$. *Then* $\mathrm{Re}[(-4\kappa/c)u_p'(z)] > \alpha$, *i.e.* $(-4\kappa/c)u_p \in \mathcal{R}_0(\alpha)$, *where* $\kappa = p + (b+1)/2$.

Proof. By Proposition 2.17, [2] we know that $4\kappa u_p'(z) = -c u_{p+1}(z)$, where $\kappa = p + (b+1)/2$. Therefore $u_{p+1}(z) = (-4\kappa/c)u_p'(z)$. Using Theorem 3.9 for $p + 1$ the inequality follows. □

Theorem 3.15. *Let* $\alpha_1 < 1, \alpha \in [0,1)$, $b \in \mathbb{R}, c \in [-37,0)$ *such that* $8(1-\alpha)\kappa + c \geq 0$. *If we have* $f \in \mathcal{R}(\alpha_1)$, *then the Bessel transform* $zu_p(z) * f(z) \in \mathcal{R}(\gamma)$, *where* $\gamma = 1 - 2(1-\alpha_1)(1-\alpha)$.

Proof. Using Theorem 3.9, we have that $u_p \in \mathcal{P}_0(\alpha)$, therefore the proof is the same like that of Theorem 3.1. □

As an immediate consequence of Theorem 3.15, we have for $\gamma = 0$.

Corollary 3.16. *Under the hypotheses of Theorem* 3.15 *with* $\gamma = 0$ *we have that if* $f(z) \in \mathcal{R}(\alpha_1)$, *then* $zu_p(z) * f(z) \in \mathcal{R}(0)$, *where* $\alpha_1 = (1-2\alpha)/(2-2\alpha)$.

Taking $\alpha = 0$ in the above Corollary, we have the next result.

Corollary 3.17. *Let* $\kappa \geq -c/4$, *where* $b \in \mathbb{R}$, $c \in [-37,0)$. *If we have* $f \in \mathcal{R}(1/2)$, *then* $zu_p(z) * f(z) \in \mathcal{R}(0)$.

It is easy to verify that the results of Corollary 3.16, Corollary 3.17 for $c \in [-37,0)$ are better than the results of Corollary 3.2, Corollary 3.3. The situation is the same for Theorem 3.15 and Theorem 3.1. So in certain cases the "method of sequences" is better than the method of differential subordinations. Note that similar results as in this paper for Gaussian and confluent hypergeometric functions may be found in [11, 12].

4. PARTICULAR CASES

We end this paper with a particular case of all the Theorems and Corollaries. Taking $b = 1$, $c = -1$ in (1.16) and (1.17) we obtain the modified Bessel function of the first kind of p order ([15], p.77)

$$I_p(z) = \mathrm{i}^{-p} J_p(\mathrm{i}z) = \sum_{n\geq 0} \frac{1}{n!\ \Gamma(n+p+1)} \left(\frac{z}{2}\right)^{2n+p},\ z \in \mathbb{C} \tag{4.1}$$

and the normalized and modified Bessel function of the first kind of p order

$$M_p(z) = 2^p \Gamma(p+1) z^{-p/2} I_p(z^{1/2}),\ z \in \mathbb{C}. \tag{4.2}$$

Proposition 4.3. For the function M_p using Theorem 2.1, 2.4, 2.6, 3.1, 3.15, Corollary 3.2, 3.3, 3.16 and 3.17, we obtain the following results:

1° If for $\alpha \in [0,1)$ we have that $4(1-\alpha)p \geq 4\alpha^2 - 4\alpha - 1$, then for $\alpha \geq 1/2$ we have $M_p \in \mathcal{H}^\infty$; and for $\alpha \in [0,1/2)$ we have $M_p \in \mathcal{H}^{1/(1-2\alpha)}$.
2° If $p \geq -5/4$, then $M_p \in \mathcal{H}^\infty$.
3° If $16p + 20 \geq \sqrt{17}$ and $f \in \mathcal{R}$, then the convolution $zM_p(z) * f(z)$ is in $\mathcal{H}^\infty \cap \mathcal{R}$.
4° Suppose that $\alpha_1 < 1$, $4p\sqrt{1-2\alpha} \geq 1-\alpha$, where $\alpha \in [0,1)\setminus\{1/2\}$. If we have $f \in \mathcal{R}(\alpha_1)$ then the Bessel transform $zM_p(z)*f(z) \in \mathcal{R}(\gamma)$, where $\gamma = 1-2(1-\alpha_1)(1-\alpha)$. For $\gamma = 0$, we have that if $f \in \mathcal{R}(\alpha_1)$, then

$zM_p(z) * f(z) \in \mathcal{R}(0)$, where $\alpha_1 = (1-2\alpha)/(2-2\alpha)$. More particulary (for $\alpha = 0$), if $p \geq 1/4$ and $f \in \mathcal{R}(1/2)$, then $zM_p(z) * f(z) \in \mathcal{R}(0)$.

5° Suppose that $\alpha_1 < 1$, $8(1-\alpha)(p+1) \geq 1$, where $\alpha \in [0,1)$. If we have $f \in \mathcal{R}(\alpha_1)$ then the Bessel transform $zM_p(z) * f(z) \in \mathcal{R}(\gamma)$, where $\gamma = 1 - 2(1-\alpha_1)(1-\alpha)$. For $\gamma = 0$, we have that if $f \in \mathcal{R}(\alpha_1)$, then $zM_p(z) * f(z) \in \mathcal{R}(0)$, where $\alpha_1 = (1-2\alpha)/(2-2\alpha)$. More particulary (for $\alpha = 0$), if $p \geq -7/8$ and $f \in \mathcal{R}(1/2)$, then $zM_p(z) * f(z) \in \mathcal{R}(0)$.

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