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PENCILS OF STRAIGHT LINES IN LOGARITHMIC POTENTIALS

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Abstract. The aim of the planar inverse problem of dynamics is to find the potentials under whose action a material point of unit mass, with appropriate initial conditions, describes the curves in a given family. We solve the following special problem: determine the finite Borel measures, with support in the unit circle, whose logarithmic potentials give rise to a family of lines passing through a given point.

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1. INTRODUCTION

The goal of the classical inverse problem of dynamics is to find the planar potentials $V = V(x,y)$ creating preassigned families of orbits, traced by a material point of unit mass. Reviews of this and of other versions of the inverse problem can be found in [10], [6] and [1].

The equations governing the motion of the particle are

$$\ddot{x} = -V_x \quad \ddot{y} = -V_y. \tag{1}$$

The very simple families of straight lines have been considered only recently. The interest in such families was raised by the fact that isolated straight line solutions have been found in galactic models by Contopoulos and Zikides [9] and by Caranicolas and Innanen [8]. Straight lines appear also in the Hénon-Heiles model [12] (van der Merwe [13], Antonov and Timoshkova [3]). Some families of straight lines were studied by Grigoriadou [11] in connection with the problem of Darboux integrability.

A monoparametric family of curves

$$f\,(x,y) = c \tag{2}$$

is determined by the *slope function*

$$\gamma = \frac{f_y}{f_x}, \tag{3}$$

where the subscripts denote partial differentiation. To each f there corresponds obviously one γ and to each γ there corresponds just one monoparametric family (2). We define also the function

$$\Gamma = \gamma\gamma_x - \gamma_y \tag{4}$$

which can be expressed in terms of the derivatives of f as

$$\Gamma = \frac{2f_{xy}f_xf_y - f_{xx}f_y^2 - f_{yy}f_x^2}{f_x^3}.$$

It follows that the curvature of the orbits in (2) is given by

$$K = |\Gamma| \,/\, \left(1+\gamma^2\right)^{3/2},$$

therefore the family (2) consists of straight lines if and only if $\Gamma = 0$. In view of (4) this condition may be written as

$$\gamma\gamma_x - \gamma_y = 0. \tag{5}$$

The potentials which produce the family of straight lines (2) (for which (5) is fulfilled) satisfy the linear first order equation

$$V_x + \gamma V_y = 0. \tag{6}$$

Equation (6) was derived in [7] as a consequence of the equation of Szebehely [16], written in terms of γ and Γ in [5]; later it was obtained directly in [2]. Expressing γ from (6) and introducing its value into (5), a nonlinear partial differential equation

$$V_xV_y\left(V_{xx} - V_{yy}\right) = V_{xy}\left(V_x^2 - V_y^2\right) \tag{7}$$

was given in [7], which must be satisfied by all potentials creating (among other orbits) a family of straight lines. It is obvious that the potential will not be uniquely determined.

It can be easily checked that for a family of straight lines through a fixed point (x_0, y_0) we have $\gamma = -(x-x_0)/(y-y_0)$, and from (6) we obtain $V(x,y) = v((x-x_0)^2 + (y-y_0)^2)$, hence the potential is an arbitrary function of the distance to the point (x_0, y_0).

2. LOGARITHMIC POTENTIALS ASSOCIATED TO A BOREL MEASURE

Betsakos and Grigoriadou [4] considered the following type of inverse problem of dynamics: Given a monoparametric family of planar curves, find the finite Borel measures supported in the unit circle, whose logarithmic potentials generate the curves of the family. The problem was solved for families of straight lines through the origin or through the point $(1,0)$, as well as for the family of circles centered at the origin. In what follows we shall consider the problem for pencils of lines through an arbitrary point of the plane.

Let σ be a finite Borel measure with support in a compact set $K \subset \mathbb{C}$. The *logarithmic potential* $V_\sigma : \mathbb{C} \to (-\infty, \infty]$ is given by

$$V_\sigma(z) = \int_K \log\frac{1}{|z-\zeta|}\,\mathrm{d}\,\sigma(\zeta), \tag{8}$$

and is harmonic in the complement of its support ([14], Ch. 3).

Using a reflection principle for harmonic functions, the following theorem was proved in [4].

Theorem 1. [4] *Let σ be a finite Borel measure with compact support $K \subset \mathbb{C}$. Suppose that the logarithmic potential* (8) *generates an orbit $\alpha \in C(I)$ given by*

$$\alpha(t) = x(t) + iy(t), \quad t \in I$$

(*I being a real interval*) *that lies on a straight line ℓ. Then V_σ is locally symmetric with respect to ℓ, i. e. $V_\sigma(z) = V_\sigma(\hat{z})$ for all z in a neighbourhood of the trace $\{\alpha(t) : t \in I\}$ of α, $\hat{z}$ being the reflection of z in ℓ.*

3. MAIN RESULTS

From now on we shall consider that the finite Borel measure σ is supported in the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Proposition 1. *The form of the logarithmic potential generated by the Lebesgue measure Λ supported in $\mathbb{T}$ is*

$$\begin{aligned} &V_\Lambda(z) = 0 \text{ for } |z| < 1; \\ &V_\Lambda(z) = -2\pi \log |z| \text{ for } |z| > 1. \end{aligned} \tag{9}$$

For the Dirac measure concentrated at $z_0 \in \mathbb{T}$ we obtain

$$V_{\delta_{z_0}}(z) = \log \frac{1}{|z - z_0|}. \tag{10}$$

Proof. Jensen's formula ([15], p. 307, Theorem 15.18) states that if g with $g(0) \neq 0$ is holomorphic on a disk centered at 0 and having the radius greater than 1, and α_1, ..., α_N are the zeros of g in $\overline{\mathbb{D}}$, then

$$|g(0)| \prod_{n=1}^{N} \frac{1}{|\alpha_n|} = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \left|g(\mathrm{e}^{\mathrm{i}\theta})\right| \mathrm{d}\,\theta\right).$$

When $N = 0$, the product is considered 1. By taking $g(\zeta) = z - \zeta$, the left hand side equals $|z| / |z| = 1$ if $0 < |z| < 1$, and $|z|$ if $|z| > 1$ (for $z = 0$, $V_\sigma(0) = 0$ obviously); therefore (9) follows. The result for the Dirac measure is obtained by an easy calculation. □

The next result expresses some properties of the logarithmic potential; a) and b) appear in the proof of the basic Theorem 4 in [4].

Theorem 2. *Let V_σ be the logarithmic potential given by* (8).

a) *If D is an open disk so that $D \cap \mathbb{T} = \emptyset$ and*

$$\frac{\partial}{\partial\theta} V_\sigma(z) = 0 \text{ for each } z = r\mathrm{e}^{\mathrm{i}\theta} \in D, \tag{11}$$

then $V_\sigma(z) = 0$ for each z in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

b) *If V_σ is constant in $\mathbb{D}$, then the Borel measure σ is a constant multiple of the Lebesgue measure Λ on $\mathbb{T}$.*

c) *If the logarithmic potential V_σ is constant in $\mathbb{C} \backslash \overline{\mathbb{D}}$, it follows that $\sigma = 0$.*

Proof. The proof of part a) makes use of the fact that $\frac{\partial}{\partial\theta}V_\sigma$ can be expressed using the Poisson transform associated to the measure σ. Part b) relies on the uniqueness of the Borel measure used in the representation of a harmonic function. Part c) follows from the fact that σ must be a multiple of the Lebesgue measure, $\sigma = C \cdot \Lambda$, and from Proposition 1 $V_\sigma(z) = -2\pi C \log|z|$ in $\mathbb{C}\backslash\overline{\mathbb{D}}$; therefore if V_σ is constant in $\mathbb{C}\backslash\overline{\mathbb{D}}$, we have $C = 0$. □

We consider now the case when the potential V_σ given by (8) gives rise to a pencil of lines through z_0.

THEOREM 3. *Let D be an open disk and z_0 a point so that $D \cup \{z_0\} \subseteq \mathbb{D}$ or $D \cup \{z_0\} \subseteq \mathbb{C}\backslash\overline{\mathbb{D}}$. Let*

$$\{s_p : p \in J\} \tag{12}$$

be the family of all chords in D passing through z_0. If V_σ generates the family (12), then V_σ is constant on the connected component containing z_0, hence $\sigma = C \cdot \Lambda$. Furthermore, if $z_0 \in \mathbb{C}\backslash\overline{\mathbb{D}}$, then $\sigma = 0$, i. e. $C = 0$.

Proof. Using Theorem 1, we obtain that V_σ is locally symmetric with respect to each line supporting s_p. It results that $V_\sigma(z)$ depends only on $|z - z_0|$, as it was already shown in the Introduction; V_σ being also harmonic, we have

$$V_\sigma(z) = a \log \frac{1}{|z - z_0|} + b, \text{ for each } z \in D\backslash\{z_0\}. \tag{13}$$

The potential V_σ being bounded at $z = z_0$, it follows that $a = 0$, hence V_σ is constant on the connected component containing z_0. Applying Theorem 2, we obtain the conclusion. □

The case $z_0 \in \mathbb{T}$ is covered by Theorem 5 from [4], where z_0 was chosen equal to 1 (which is possible by means of a rotation). We shall state the theorem for arbitrary z_0.

THEOREM 4. *Let D be an open disk and consider the family* (12) *of all chords in D passing through $z_0 \in \mathbb{T}$, generated by the logarithmic potential V_σ.*

If $D \subset \mathbb{D}$, then $\sigma = C_1 \cdot \Lambda + C_2 \cdot \delta_{z_0}$, where C_1 and C_2 are constants, and δ_{z_0} is the Dirac measure concentrated at z_0.

If $D \subseteq \mathbb{C}\backslash\overline{\mathbb{D}}$, then $\sigma = C_3 \cdot \delta_{z_0}$, where C_3 is a constant.

REMARK 1. This type of inverse problem, treated here for pencils of lines, can be considered for various families of functions. In [4] it was proved that if a logarithmic potential V_σ generates each circular arc in a disk $D \subset \mathbb{C} \setminus \mathbb{T}$, then $\sigma = C \cdot \Lambda$, where C is a constant. We mention that if $D \subset \mathbb{D}$, then from Proposition 1 it follows that $V_\sigma(z) = 0$ in $\mathbb{D}$, and this potential does not produce any circle.

The problem of finding all logarithmic potentials which give rise to families of parallel lines is still open.

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