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NEW CLASSES OF SALAGEAN-TYPE MULTIVALENT HARMONIC FUNCTIONS

SİBEL YALÇIN, HAKAN BOSTANCI and METİN ÖZTÜRK

Abstract. New classes of Salagean-type multivalent harmonic functions are introduced. We give sufficient coefficient conditions for these classes. These coefficient conditions are shown to be also necessary if certain restrictions are imposed on the coefficients of these harmonic functions. Furthermore, we determine a representation theorem, inclusion relations, and distortion bounds for these functions.

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1. INTRODUCTION

A continuous function $f = u + \mathrm{i}v$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see Clunie and Sheil-Small [3]).

Denote by H the family of functions $f = h+\bar{g}$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ so that $f = h+\bar{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$.

Recently, Ahuja and Jahangiri [2] defined the class $H_p(n)$ $(p, n \in \mathbb{N} = \{1, 2, 3, ...\})$ consisting of all p-valent harmonic functions $f = h + \bar{g}$ that are sense-preserving in U, and h, g are of the form

$$(1) \qquad h(z) = z^p + \sum_{k=n+p}^{\infty} a_k\, z^k, \qquad g(z) = \sum_{k=n+p-1}^{\infty} b_k\, z^k, \quad |b_{n+p-1}| < 1.$$

For $f = h + \bar{g}$ given by (1), the modified Salagean operator of f is defined as:

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)}\,; \quad p > m, \quad m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},$$

where

$$D^m h(z) = p^m z^p + \sum_{k=n+p}^{\infty} k^m a_k z^k \quad \text{and} \quad D^m g(z) = \sum_{k=n+p-1}^{\infty} k^m b_k z^k$$

(see [4], [5]).

Also, the subclasses denote by $H_p^m(n)$ consist of harmonic functions $f_m = h + \overline{g}_m$, so that h and g_m are of the form

$$h(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad g_m(z) = (-1)^m \sum_{k=n+p-1}^{\infty} b_k z^k, \tag{2}$$

for $a_k, b_k \geq 0$, $|b_{n+p-1}| < 1$. A function f in $H_p(n)$ is said to be in the class $H_p^m(n;\lambda,\alpha)$ if

$$\mathrm{Re}\left\{(1-\lambda)\frac{D^m f(z)}{\frac{\partial^m}{\partial\theta^m} z^p} + \lambda \frac{D^{m+1} f(z)}{\frac{\partial^{m+1}}{\partial\theta^{m+1}} z^p}\right\} > \frac{\alpha}{p^{m+1}},$$

where $0 \leq \alpha < p$, $\lambda \geq 0$, $p \geq m$ and $z = re^{i\theta} \in U$.

As λ changes from 0 to 1, the family $H_p^m(n;\lambda,\alpha)$ provides a passage from the class of Salagean-type multivalent harmonic functions $H_p^m P(n;\alpha) \equiv H_p^m(n;0,\alpha)$ consisting of functions f where

$$\mathrm{Re}\left\{\frac{D^m f(z)}{\frac{\partial^m}{\partial\theta^m} z^p}\right\} > \frac{\alpha}{p^{m+1}}$$

to the class of Salagean-type multivalent harmonic functions $H_p^m Q(n;\alpha) \equiv H_p^m(n;1,\alpha)$ consisting of functions f where

$$\mathrm{Re}\left\{\frac{D^{m+1} f(z)}{\frac{\partial^{m+1}}{\partial\theta^{m+1}} z^p}\right\} > \frac{\alpha}{p^{m+1}}.$$

Finally, we define the subclass $\overline{H}_p^m(n;\lambda,\alpha) \equiv H_p^m(n;\lambda,\alpha) \cap H_p^m(n)$. The class $H_p^m(n;\lambda,\alpha)$ includes a variety of well-known subclasses of $H_p(n)$. For example, $H_p^0(n;\lambda,\alpha)$ is studied in [1].

We obtain sufficient coefficient bounds for functions in $H_p^m(n;\lambda,\alpha)$. These sufficient coefficient conditions are shown to be also necessary for functions in $\overline{H}_p^m(n;\lambda,\alpha)$. A representation theorem, inclusion properties, and distortion bounds for the class $\overline{H}_p^m(n;\lambda,\alpha)$ are also obtained.

2. REPRESENTATION THEOREM

We begin with a sufficient condition for functions in $H_p^m(n;\lambda,\alpha)$.

THEOREM 1. *Let $f = h + \bar{g}$ be given by* (1). *Then $f \in H_p^m(n;\lambda,\alpha)$ if*

$$\sum_{k=n+p}^{\infty} |\lambda k + (1-\lambda)p|\, k^m\, |a_k| + \sum_{k=n+p-1}^{\infty} |\lambda k - (1-\lambda)p|\, k^m\, |b_k| \leq p^{m+1} - \alpha \tag{3}$$

Proof. Using the fact that $\mathrm{Re}\zeta \geq 0$ if and only if $|1+\zeta| \geq |1-\zeta|$ in U, it suffices to show that

$$\left|p^{m+1} - \alpha + p^{m+1} w\right| \geq \left|p^{m+1} + \alpha - p^{m+1} w\right|,$$

where

$$w(z) = (1-\lambda)\frac{D^m f(z)}{\frac{\partial^m}{\partial\theta^m} z^p} + \lambda \frac{D^{m+1} f(z)}{\frac{\partial^{m+1}}{\partial\theta^{m+1}} z^p} \cdot$$

Substituting for h and g in w, we obtain

$$\begin{aligned} \left|p^{m+1} - \alpha + p^{m+1} w\right| \geq\ & 2p^{m+1} - \alpha - \sum_{k=n+p}^{\infty} |p + \lambda(k-p)|\, k^m |a_k|\, |z|^{k-p} \\ & - \sum_{k=n+p-1}^{\infty} |p - \lambda(k+p)|\, k^m b_k\, |z|^{k-p} \end{aligned}$$

and

$$\begin{aligned} \left|p^{m+1} + \alpha - p^{m+1} w\right| \leq\ & \alpha + \sum_{k=n+p}^{\infty} |p + \lambda(k-p)|\, k^m |a_k|\, |z|^{k-p} \\ & + \sum_{k=n+p-1}^{\infty} |p - \lambda(k+p)|\, k^m b_k\, |z|^{k-p}. \end{aligned}$$

These two inequalities in conjunction with the required condition (3) yield

$$\begin{aligned} & \left|p^{m+1} - \alpha + p^{m+1} w\right| - \left|p^{m+1} + \alpha - p^{m+1} w\right| \\ \geq\ & 2\Bigg[p^{m+1} - \alpha - \sum_{k=n+p}^{\infty} |\lambda k + (1-\lambda)p|\, k^m |a_k| \\ & + \sum_{k=n+p-1}^{\infty} |\lambda k - (1-\lambda)p|\, k^m |b_k|\Bigg] \geq 0. \end{aligned}$$

The coefficient bound (3) gave in Theorem 1 is sharp for the function

$$f(z) = z^p + \sum_{k=n+p}^{\infty} \frac{x_k}{|\lambda k + (1-\lambda)p|\, k^m}\ z^k + \sum_{k=n+p-1}^{\infty} \frac{\bar{y}_k}{|\lambda k + (1-\lambda)p|\, k^m}\ \bar{z}^k,$$

where $\sum_{k=n+p}^{\infty} |x_k| + \sum_{k=n+p-1}^{\infty} |y_k| = p^{m+1} - \alpha$. □

THEOREM 2. *Let* $f_m = h + \overline{g}_m$ *be given by* (2). *Then* $f_m \in \overline{H}_p^m(n;\lambda,\alpha)$ *if and only if*

$$\sum_{k=n+p}^{\infty} |\lambda k + (1-\lambda)p|\, k^m a_k + \sum_{k=n+p-1}^{\infty} |\lambda k - (1-\lambda)p|\, k^m b_k \leq p^{m+1} - \alpha. \tag{4}$$

Proof. In view of Theorem 1, we only need to prove the only if part of the theorem, since $\overline{H}_p^m(n;\lambda,\alpha) \subset H_p^m(n;\lambda,\alpha)$.

If $f_m \in \bar{H}_p^m(n;\lambda,\alpha)$ then, for $z = re^{i\theta}$ in U we get

$$\begin{aligned}
& \operatorname{Re}\left\{(1-\lambda)\frac{D^m f_m(z)}{\frac{\partial^m}{\partial\theta^m}z^p} + \lambda\frac{D^{m+1} f_m(z)}{\frac{\partial^{m+1}}{\partial\theta^{m+1}}z^p}\right\} \\
= & \operatorname{Re}\left\{\frac{(1-\lambda)}{p^m}\left(\frac{D^m h(z) + (-1)^m\overline{D^m g_m(z)}}{i^m z^p}\right)\right. \\
& \left. + \frac{\lambda}{p^{m+1}}\left(\frac{D^{m+1} h(z) - (-1)^m\overline{D^{m+1} g_m(z)}}{i^{m+1} z^p}\right)\right\} \\
\geq & 1 - \frac{1}{p^{m+1}}\sum_{k=n+p}^{\infty}|\lambda k + (1-\lambda)p|\, k^m a_k r^{k-p} \\
& - \frac{1}{p^{m+1}}\sum_{k=n+p-1}^{\infty}|\lambda k - (1-\lambda)p|\, k^m b_k r^{k-p} \\
\geq & \frac{\alpha}{p^{m+1}}.
\end{aligned}$$

This inequality must hold for all $z \in U$. In particular, letting $z = r \to 1$, it yields the required condition (4). □

As special cases of Theorem 2, we obtain the following two corollaries:

Corollary 1. *$f_m = h + \overline{g}_m \in \overline{H}_p^m P(n;\alpha) \equiv H_p^m P(n;\alpha) \cap H_p^m(n)$ if and only if*

$$\sum_{k=n+p}^{\infty}\frac{pk^m}{p^{m+1}-\alpha}a_k + \sum_{k=n+p-1}^{\infty}\frac{pk^m}{p^{m+1}-\alpha}b_k \leq 1.$$

Corollary 2. *$f_m = h + \overline{g}_m \in \overline{H}_p^m Q(n;\alpha) \equiv H_p^m Q(n;\alpha) \cap H_p^m(n)$ if and only if*

$$\sum_{k=n+p}^{\infty}\frac{k^{m+1}}{p^{m+1}-\alpha}a_k + \sum_{k=n+p-1}^{\infty}\frac{k^{m+1}}{p^{m+1}-\alpha}b_k \leq 1.$$

Now, we determine a representation theorem for functions in $\overline{H}_p^m(n;\lambda,\alpha)$.

Theorem 3. *$f_m = h + \overline{g}_m \in \overline{H}_p^m(n;\lambda,\alpha)$ if and only if f_m can be expressed as*

$$f_m(z) = X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_m}(z),$$

where $h_p(z) = z^p$, $h_k(z) = z^p - \frac{p^{m+1}-\alpha}{|\lambda k+(1-\lambda)p|k^m}z^k$, $(k = n+p, n+p+1, \ldots)$, $g_{k_m}(z) = z^p + (-1)^m\frac{p^{m+1}-\alpha}{|\lambda k-(1-\lambda)p|k^m}\bar{z}^k$, $(k = n+p-1, n+p, \ldots)$, $X_p \geq 0$, $Y_{n+p-1} \geq 0$, $X_p + \sum_{k=n+p}^{\infty} X_k + \sum_{k=n+p-1}^{\infty} Y_k = 1$, and $X_k \geq 0$, $Y_k \geq 0$, for $k = n+p, n+p+1, \ldots$.

Proof. For functions f_m of the form (5) we have

$$\begin{aligned} f_m(z) &= X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_m}(z) \\ &= z^p - \sum_{k=n+p}^{\infty} \frac{p^{m+1}-\alpha}{|\lambda k + (1-\lambda)p|\, k^m} X_k z^k \\ &\quad +(-1)^m \sum_{k=n+p-1}^{\infty} \frac{p^{m+1}-\alpha}{|\lambda k - (1-\lambda)p|\, k^m} Y_k \bar{z}^k. \end{aligned}$$

Consequently, $f_m \in \overline{H}_p^m(n;\lambda,\alpha)$, since by (4), we have

$$\begin{aligned} &\sum_{k=n+p}^{\infty} |\lambda k + (1-\lambda)p|\, k^m a_k + \sum_{k=n+p-1}^{\infty} |\lambda k - (1-\lambda)p|\, k^m b_k \\ =& \sum_{k=n+p}^{\infty} |\lambda k + (1-\lambda)p|\, k^m \frac{p^{m+1}-\alpha}{|\lambda k + (1-\lambda)p|\, k^m} |X_k| \\ &+ \sum_{k=n+p-1}^{\infty} |\lambda k - (1-\lambda)p|\, k^m \frac{p^{m+1}-\alpha}{|\lambda k - (1-\lambda)p|\, k^m} |Y_k| \\ =& (p^{m+1}-\alpha)\Big(\sum_{k=n+p}^{\infty} |X_k| + \sum_{k=n+p-1}^{\infty} |Y_k|\Big) = (p^{m+1}-\alpha)(1-X_p) \\ \leq& \ p^{m+1}-\alpha. \end{aligned}$$

Conversely, suppose $f_m \in \overline{H}_p^m(n;\lambda,\alpha)$. Letting $X_p = 1 - \sum_{k=n+p}^{\infty} X_k - \sum_{k=n+p-1}^{\infty} Y_k$, where $X_k = \frac{|\lambda k+(1-\lambda)p| k^m}{p^{m+1}-\alpha} a_k$, and $Y_k = \frac{|\lambda k-(1-\lambda)p| k^m}{p^{m+1}-\alpha} b_k$, we obtain the required representation, since

$$\begin{aligned} f_m(z) &= z^p - \sum_{k=n+p}^{\infty} a_k z^k + (-1)^m \sum_{k=n+p-1}^{\infty} b_k \bar{z}^k \\ &= z^p - \sum_{k=n+p}^{\infty} \frac{(p^{m+1}-\alpha)X_k}{|\lambda k + (1-\lambda)p|\, k^m} z^k \\ &\quad +(-1)^m \sum_{k=n+p-1}^{\infty} \frac{(p^{m+1}-\alpha)Y_k}{|\lambda k - (1-\lambda)p|\, k^m} \bar{z}^k \\ &= z^p - \sum_{k=n+p}^{\infty} (z^p - h_k(z))X_k - \sum_{k=n+p-1}^{\infty} (z^p - g_{k_m}(z))Y_k \\ &= \left(1 - \sum_{k=n+p}^{\infty} X_k - \sum_{k=n+p-1}^{\infty} Y_k\right) z^p \end{aligned}$$

$$
\begin{aligned}
& + \sum_{k=n+p}^{\infty} h_k(z) X_k + \sum_{k=n+p-1}^{\infty} g_{k_m}(z) Y_k \\
= \; & X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_m}(z). \qquad \square
\end{aligned}
$$

The inclusion relations between the classes for the different values of λ are not so obvious. Now we discuss the inclusion relations between the above mentioned classes.

THEOREM 4. *For $n \in \mathbb{N}$ and $0 \le \alpha < p$, we have:*
(i) $\overline{H}_p^m Q(n;\alpha) \subset \overline{H}_p^m P(n;\alpha)$,
(ii) $\overline{H}_p^m Q(n;\alpha) \subset \overline{H}_p^m (n;\lambda,\alpha)$, $0 < \lambda \le 1$,
(iii) $\overline{H}_p^m (n;\lambda,\alpha) \subset \overline{H}_p^m Q(n;\alpha)$, $\lambda \ge 1$.

Proof. (i) In view of Corollaries 1 and 2, since

$$
\sum_{k=n+p}^{\infty} p k^m a_k + \sum_{k=n+p-1}^{\infty} p k^m b_k \le \sum_{k=n+p}^{\infty} k^{m+1} a_k + \sum_{k=n+p-1}^{\infty} k^{m+1} b_k \le p^{m+1} - \alpha,
$$

the result follows.

(ii) For $0 \le \lambda < 1$, we have

$$
\begin{aligned}
& \sum_{k=n+p}^{\infty} |\lambda k + (1-\lambda)p| \, k^m a_k + \sum_{k=n+p-1}^{\infty} |\lambda k - (1-\lambda)p| \, k^m b_k \\
= \; & \sum_{k=n+p}^{\infty} [\lambda(k-p) + p] k^m a_k + \sum_{k=n+p-1}^{\infty} [\lambda(k+p) - p] k^m b_k \\
\le \; & \sum_{k=n+p}^{\infty} k^{m+1} a_k + \sum_{k=n+p-1}^{\infty} k^{m+1} b_k \le p^{m+1} - \alpha
\end{aligned}
$$

by Corollary 2. Thus, (ii) is obtained from Theorem 2.

(iii) If $\lambda \ge 1$, then, by Theorem 2,

$$
\begin{aligned}
& \sum_{k=n+p}^{\infty} k^{m+1} a_k + \sum_{k=n+p-1}^{\infty} k^{m+1} b_k \\
\le \; & \sum_{k=n+p}^{\infty} [\lambda(k-p) + p] k^m a_k + \sum_{k=n+p-1}^{\infty} [\lambda(k+p) - p] k^m b_k \\
\le \; & \sum_{k=n+p}^{\infty} |\lambda k + (1-\lambda)p| \, k^m a_k + \sum_{k=n+p-1}^{\infty} |\lambda k - (1-\lambda)p| \, k^m b_k \\
\le \; & p^{m+1} - \alpha.
\end{aligned}
$$

Therefore, (iii) is obtained from Corollary 2. $\square$

Finally, we give a distortion theorem for functions in $\overline{H}_p^m(n;\lambda,\alpha)$, which leads to a covering result for this family.

THEOREM 5. *If* $f_m \in \overline{H}_p^m(n;\lambda,\alpha)$, $\lambda \geq 1$ *and* $|z| = r < 1$, *then*

$$|f_m(z)| \leq (1 + b_{n+p-1} r^{n-1}) r^p + \left(\frac{p^{m+1} - \alpha}{(\lambda n + p)(n+p)^m} - \frac{[\lambda(n+2p-1) - p](n+p-1)^m}{(\lambda n + p)(n+p)^m} b_{n+p-1} \right) r^{n+p}$$

and

$$|f_m(z)| \geq (1 - b_{n+p-1} r^{n-1}) r^p - \left(\frac{p^{m+1} - \alpha}{(\lambda n + p)(n+p)^m} - \frac{[\lambda(n+2p-1) - p](n+p-1)^m}{(\lambda n + p)(n+p)^m} b_{n+p-1} \right) r^{n+p}.$$

Proof. We prove the left hand side inequality for $|f_m|$. The proof for the right hand side inequality can be done using similar arguments.

Let $f_m \in \overline{H}_p^m(n;\lambda,\alpha)$, then by Theorem 2, we obtain:

$$|f_m(z)| = \left| z^p + (-1)^m b_{n+p-1}\, \overline{z}^{\,n+p-1} + \sum_{k=n+p}^{\infty} \left(a_k z^k + (-1)^m b_k\, \overline{z}^{\,k} \right) \right|$$

$$\geq r^p - b_{n+p-1} r^{n+p-1} - \frac{p^{m+1} - \alpha}{(\lambda n + p)(n+p)^m} \sum_{k=n+p}^{\infty} \left(\frac{p + \lambda n}{p^{m+1} - \alpha} a_k + \frac{p + \lambda n}{p^{m+1} - \alpha} b_k \right) (n+p)^m r^k$$

$$\geq r^p - b_{n+p-1} r^{n+p-1} - \frac{p^{m+1} - \alpha}{(\lambda n + p)(n+p)^m} \sum_{k=n+p}^{\infty} \left(\frac{\lambda(k-p) + p}{p^{m+1} - \alpha} a_k + \frac{\lambda(k+p) - p}{p^{m+1} - \alpha} b_k \right) k^m r^k$$

$$\geq (1 - b_{n+p-1} r^{n-1}) r^p - \frac{p^{m+1} - \alpha}{(\lambda n + p)(n+p)^m} \left[1 - \frac{[\lambda(n+2p-1) - p](n+p-1)^m}{p^{m+1} - \alpha} b_{n+p-1} \right] r^{n+p}$$

$$\geq (1 - b_{n+p-1} r^{n-1}) r^p - \left(\frac{p^{m+1} - \alpha}{(\lambda n + p)(n+p)^m} - \frac{[\lambda(n+2p-1) - p](n+p-1)^m}{(\lambda n + p)(n+p)^m} b_{n+p-1} \right) r^{n+p}. \quad \square$$

The following covering result follows from the left hand side inequality in Theorem 5.

COROLLARY 3. *If* $f_m \in \overline{H}_p^m(n;\lambda,\alpha)$, $\lambda \geq 1$, *then the set*

$$\left\{ w : |w| < \frac{(\lambda n+p)(n+p)^m - p^{m+1} + \alpha - [(p+\lambda n)(n+p)^m + [\lambda(n+2p-1) - p](n+p-1)^m] b_{n+p-1}}{(\lambda n+p)(n+p)^m} \right\}$$

is included in $f_m(U)$.

Using arguments similar to those given in the proof of Theorem 5, we obtain the following two theorems.

THEOREM 6. *If* $f_m \in \overline{H}_p^m P(n;\alpha)$, *then*

$$|f_m(z)| \le (1 + b_{n+p-1} r^{n-1}) r^p + \left(\frac{p^{m+1} - \alpha}{p(n+p)^m} + \frac{(n+p-1)^m}{(n+p)^m} b_{n+p-1} \right) r^{n+p},$$

and

$$|f_m(z)| \ge (1 - b_{n+p-1} r^{n-1}) r^p - \left(\frac{p^{m+1} - \alpha}{p(n+p)^m} + \frac{(n+p-1)^m}{(n+p)^m} b_{n+p-1} \right) r^{n+p}.$$

THEOREM 7. *If* $f_m \in \overline{H}_p^m Q(n;\alpha)$, *then*

$$|f_m(z)| \le (1 + b_{n+p-1} r^{n-1}) r^p + \left(\frac{p^{m+1} - \alpha}{(n+p)^{m+1}} - \frac{(n+p-1)^{m+1}}{(n+p)^{m+1}} b_{n+p-1} \right) r^{n+p},$$

and

$$|f_m(z)| \ge (1 - b_{n+p-1} r^{n-1}) r^p - \left(\frac{p^{m+1} - \alpha}{(n+p)^{m+1}} - \frac{(n+p-1)^{m+1}}{(n+p)^{m+1}} b_{n+p-1} \right) r^{n+p}.$$

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Department of Mathematics
Faculty of Science
University of Uludag
Görükle 16059, Bursa, Turkey
E-mail: skarpuz@uludag.edu.tr
E-mail: hbostanci@uludag.edu.tr
E-mail: ometin@uludag.edu.tr