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A SHARP CRITERION FOR STARLIKENESS

RÓBERT SZÁSZ

Abstract. In this paper we have obtained a simple sufficient condition for the starlikeness of analytic functions defined in the unit disc $U = \{z \in C : |z| < 1\}$.

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1. INTRODUCTION

Let A be the class of functions which are analytic in the unit disc $U = \{z \in \mathbb{C} \mid |z| < 1\}$ and have the form $f(z) = z + a_2z^2 + a_3z^3 + \dots$. We denote by S^* the subclass of A consist of starlike (univalent) functions in U, i.e.

$$S^* = \left\{ f \in A : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0, z \in U \right\}.$$

In [2] P.T. Mocanu has proved that if $f \in A$ and $\mathrm{Re}(f'(z) + \frac{1}{2}f''(z)) > 0$, $z \in U$, then f belongs to S^*. In this paper we will determine the biggest value of c for which the condition $f \in A$, $\mathrm{Re}(f'(z) + \frac{1}{2}f''(z)) > -c$, $z \in U$ implies the starlikeness of the function f.

2. PRELIMINARIES

In [2] the author used the method of differential subordination. Now we apply the duality principle for convolution. Let $f(z) = \sum_{n=0}^{\infty} a_nz^n$ and $g(z) = \sum_{n=0}^{\infty} b_nz^n$ be two analytic function in U. The Hadamard product (convolution) of f and g is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_nb_nz^n.$$

Let A_0 be the class of the analytic functions f in U which have the property $f(0) = 1$ and let $\mathcal{P}$ be the subclass of A_0 defined by the equality $\mathcal{P} = \{f \in A_0 : \mathrm{Re}(f(z)) > 0, z \in U\}$. For $V \subset A_0$ the dual V^* is the set of functions $g \in A_0$ such that $(f * g)(z) \neq 0$ for every $f \in V$ and $z \in U$. We define the functions h_T in A by

$$h_T(z) = \frac{1}{1 + \mathrm{i}T}\left[\mathrm{i}T\frac{z}{1-z} + \frac{z}{(1-z)^2}\right], \quad T \in R.$$

We need the following theorems, which have been established in [5].

THEOREM 1. [5, p. 23] (The duality principle) *The dual of the class* $\mathcal{P}$ *satisfies the equality* $\mathcal{P}^* = \{f \in A_0 | \mathrm{Re}(f(z)) > \frac{1}{2}\ , z \in U\}$.

THEOREM 2. [5, p. 94] *A function $f \in A$ is in the class S^* of the starlike functions if and only if $\frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0$ for every $T \in R$ and $z \in U$.*

THEOREM 3. *If the function $f : [0,1] \to R_+^*$ is increasing on $[0,1]$ and $a > 0$, then*

$$\int_0^1 \frac{x^{a+1}}{f(x)} \mathrm{d}x \leq \frac{a+1}{a+2} \int_0^1 \frac{x^a}{f(x)} \mathrm{d}x.$$

Proof. We have:

$$\int_0^1 \frac{x^{a+1}}{f(x)} \mathrm{d}x = \frac{a+1}{a+2} \int_0^1 \frac{x^a}{f\left(x^{\frac{a+1}{a+2}}\right)} \mathrm{d}x \leq \frac{a+1}{a+2} \int_0^1 \frac{x^a}{f(x)} \mathrm{d}x.$$

3. MAIN RESULT

THEOREM 4. *The biggest value of c for which the condition*

$$f \in A, \quad \mathrm{Re}\left(f'(z) + \frac{1}{2} z f''(z)\right) > -c, \quad z \in U \tag{1}$$

implies the starlikness of the function f is $c = \frac{3 - 4\ln 2}{4 \ln 2 - 2}$. The condition (1) does not imply even the univalence of the function f for a bigger value of c.

Proof. The condition (2) implies that $\frac{c + f'(z) + \frac{1}{2} z f''(z)}{1+c} \in \mathcal{P}$ and from the Herglotz representation theorem we get

$$\frac{c + f'(z) + \frac{1}{2} z f''(z)}{1+c} = 1 + 2 \sum_{n=1}^{\infty} z^n \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}\mu(t), \tag{2}$$

where μ is a probability measure on $[0, 2\pi]$. If the function f has the development $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then from (3) it follows that

$$a_n = \frac{4(1+c)}{n(n+1)} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}(n-1)t} \mathrm{d}\mu(t) \quad \text{for} \quad n \geq 2$$

and

$$f(z) = z + 4(1+c) \sum_{n=2}^{\infty} \frac{z^n}{n(n+1)} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}(n-1)t} \mathrm{d}\mu(t).$$

According to Theorem 2, the function $f \in A$ is starlike with respect to 0 if and only if

$$\frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0, \quad z \in U, \quad T \in R. \tag{3}$$

The function h_T has the development $h_T(z) = z + \sum_{n=2}^{\infty} \frac{n + \mathrm{i}T}{1 + \mathrm{i}T} z^n$ and the convolution of the functions f and h_T can be rewritten in the following form:

$$\frac{f(z)}{z} * \frac{h_T(z)}{z} = \left(1 + 4(1+c)\sum_{n=1}^{\infty} \frac{z^n}{(n+1)(n+2)} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}\mu(t)\right) * \left(1 + \sum_{n=1}^{\infty} \frac{n+1+\mathrm{i}T}{1+\mathrm{i}T} z^n\right) = \left(1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}\mu(t)\right) * \left(1 + 2(1+c)\sum_{n=1}^{\infty} \frac{n+1+\mathrm{i}T}{(1+\mathrm{i}T)(n+1)(n+2)} z^n\right). \tag{4}$$

Because $1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}nt}\mathrm{d}\mu(t) \in \mathcal{P}$, from Theorem 1 and using (3), (4), we get that the function f belongs to S^* if and only if

$$\mathrm{Re}\left(1 + 2(1+c)\sum_{n=1}^{\infty} \frac{n+1+\mathrm{i}T}{(1+\mathrm{i}T)(n+1)(n+2)} z^n\right) > \frac{1}{2},\ z \in U,\ T \in R,$$

or, equivalently

$$\mathrm{Re}\left(\frac{1}{4(1+c)} + \sum_{n=1}^{\infty} \frac{n+1+\mathrm{i}T}{(1+\mathrm{i}T)(n+1)(n+2)} z^n\right) > 0, \quad z \in U, \quad T \in R. \tag{5}$$

In order to determine the biggest value of c for which the starlikeness condition (5) is valid we must compute

$$m = \inf_{\substack{z \in U \\ T \in R}} \left(\mathrm{Re}\left(\frac{1}{4(1+c)} + \sum_{n=1}^{\infty} \frac{n+1+\mathrm{i}T}{(1+\mathrm{i}T)(n+1)(n+2)} z^n\right)\right).$$

It is easy to observe that:

$$m = \inf_{\substack{\theta \in (0,2\pi) \\ T \in R}} \left(\mathrm{Re}\left(\frac{1}{4(1+c)} + \sum_{n=1}^{\infty} \frac{n+1+\mathrm{i}T}{(1+\mathrm{i}T)(n+1)(n+2)} \mathrm{e}^{\mathrm{i}n\theta}\right)\right). \tag{6}$$

We introduce the notation:

$$M(\theta, T) = \mathrm{Re}\left(\sum_{n=1}^{\infty} \frac{n+1+\mathrm{i}T}{(1+\mathrm{i}T)(n+1)(n+2)} \mathrm{e}^{\mathrm{i}n\theta}\right)$$

and after some calculations we get

$$M(\theta, T) = \mathrm{Re}\left(\frac{1}{1+T^2}\left(\sum_{n=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}n\theta}}{n+2} - \mathrm{i}T\sum_{n=1}^{\infty} \frac{n\mathrm{e}^{\mathrm{i}n\theta}}{(n+1)(n+2)} + T^2\sum_{n=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}n\theta}}{(n+1)(n+2)}\right)\right). \tag{7}$$

Our aim is to show that $M(\theta,T)\geq 1-\ln 2$, $\theta\in(0,2\pi)$, $T\in\mathbb{R}$. From the identities

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}n\theta}}{n+\beta} &= \int_0^1 t^{\beta}\frac{\mathrm{e}^{\mathrm{i}\theta}-t}{1+t^2-2t\cos\theta}\mathrm{d}t,\\ \sum_{n=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}n\theta}}{(n+\beta)(n+\gamma)} &= \int_0^1\int_0^1 t^{\beta}x^{\gamma}\frac{e^{\mathrm{i}\theta}-tx}{1+t^2x^2-2tx\cos\theta}\mathrm{d}t\mathrm{d}x,\end{aligned}$$

where $\theta\in(0,2\pi)$, $\beta,\gamma\in(0,+\infty)$, it follows that

$$\begin{aligned}M(\theta,T) &= \frac{1}{1+T^2}\bigg(\int_0^1 t^2\frac{\cos\theta-t}{1+t^2-2t\cos\theta}\mathrm{d}t\\ &+T\sin\theta\int_0^1\int_0^1 t^2\frac{(1-x)(1-t^2x)}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\mathrm{d}t\mathrm{d}x\\ &+T^2\int_0^1\int_0^1 t^2x\frac{\cos\theta-tx}{1+t^2x^2-2tx\cos\theta}\mathrm{d}t\mathrm{d}x\bigg).\end{aligned}$$

Using the identities

$$\begin{aligned}&\int_0^1\frac{t^2}{1+t}\mathrm{d}t-\int_0^1\int_0^1\frac{t^2x}{1+tx}\mathrm{d}t\mathrm{d}x=\int_0^1\int_0^1 t^2\frac{1-x}{(1+t)(1+tx)}\mathrm{d}t\mathrm{d}x,\\ &\int_0^1 t^2\frac{\cos\theta-t}{1+t^2-2t\cos\theta}\mathrm{d}t+\int_0^1\frac{t^2}{1+t}\mathrm{d}t\\ &\quad=(1+\cos\theta)\int_0^1 t^2\frac{1-t}{(1+t)(1+t^2-2t\cos\theta)}\mathrm{d}t,\\ &\int_0^1\int_0^1 t^2x\frac{\cos\theta-tx}{1+t^2x^2-2tx\cos\theta}\mathrm{d}t\mathrm{d}x+\int_0^1\int_0^1\frac{t^2x}{1+tx}\mathrm{d}t\mathrm{d}x\\ &\quad=(1+\cos\theta)\int_0^1\int_0^1 t^2x\frac{1-tx}{(1+t^2x^2-2tx\cos\theta)(1+xt)}\mathrm{d}t\mathrm{d}x,\end{aligned}$$

we obtain:

$$\begin{aligned}M(\theta,T) &= -\int_0^1\frac{t^2}{1+t}\mathrm{d}t+\frac{1}{1+T^2}\bigg(T^2\int_0^1\int_0^1 t^2\frac{1-x}{(1+t)(1+tx)}\mathrm{d}t\mathrm{d}x\\ &+(1+\cos\theta)\int_0^1 t^2\frac{1-t}{(1+t)(1+t^2-2t\cos\theta)}\mathrm{d}t\\ &+T\sin\theta\int_0^1\int_0^1 t^2\frac{(1-t^2x)(1-x)}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\mathrm{d}t\mathrm{d}x\bigg)\\ &+\frac{T^2}{1+T^2}(1+\cos\theta)\int_0^1\int_0^1 t^2x\frac{1-tx}{(1+t^2x^2-2tx\cos\theta)(1+xt)}\mathrm{d}t\mathrm{d}x.\end{aligned}$$

Let us introduce the notations:

$$\begin{aligned} L_1(\theta,T) &= T^2\int_0^1\int_0^1 t^2\frac{1-x}{(1+t)(1+tx)}\mathrm{d}t\mathrm{d}x \\ &\quad +(1+\cos\theta)\int_0^1 t^2\frac{1-t}{(1+t)(1+t^2-2t\cos\theta)}\mathrm{d}t \\ &\quad +T\sin\theta\int_0^1\int_0^1 t^2\frac{(1-t^2x)(1-x)}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\mathrm{d}t\mathrm{d}x \end{aligned}$$

and

$$L_2(\theta,T)=\frac{T^2}{1+T^2}(1+\cos\theta)\int_0^1\int_0^1 t^2x\frac{1-tx}{(1+t^2x^2-2tx\cos\theta)(1+xt)}\mathrm{d}t\mathrm{d}x.$$

A simple calculation shows that $M(\theta,T)=L_1(\theta,T)+L_2(\theta,T)+\frac{1}{2}-\ln 2$. We observe that

$$L_2(\theta,T)\geq 0,\quad \theta\in(0,2\pi),\quad T\in R \tag{8}$$

and we will prove the inequality

$$L_1(\theta,T)\geq 0,\quad \theta\in\left(0,\frac{17\pi}{12}\right),\quad T\in R. \tag{9}$$

We discuss the inequality (9) in two cases. If $\theta\in(0,\pi)$, then (9) is valid. The expression $L_1(\theta,T)$ is a polynomial of degree two in T. The coefficient of T^2 is positive and so, if we prove that the discriminant of $L_1(\theta,T)$ is negative, it will follow (9). The discriminant of $L_1(\theta,T)$ is:

$$\begin{aligned} \Delta(\theta) &= \left(\sin\theta\int_0^1\int_0^1 t^2\frac{(1-t^2x)(1-x)}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\mathrm{d}t\mathrm{d}x\right)^2 \\ &- 4(1+\cos\theta)\int_0^1 t^2\frac{1-t}{(1+t)(1+t^2-2t\cos\theta)}\mathrm{d}t\int_0^1\int_0^1 t^2\frac{1-x}{(1+t)(1+tx)}\mathrm{d}t\mathrm{d}x, \end{aligned}$$

which can be rewritten in the form

(10)

$$\begin{aligned} \Delta(\theta) &= 4\cos^2\frac{\theta}{2} \\ &\cdot\left(\left(\sin\frac{\theta}{2}\int_0^1\int_0^1 t^2\frac{(1-t^2x)(1-x)}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\mathrm{d}t\mathrm{d}x\right)^2\right. \\ &\left. - 2\int_0^1 t^2\frac{1-t}{(1+t)(1+t^2-2t\cos\theta)}\mathrm{d}t\int_0^1\int_0^1 t^2\frac{1-x}{(1+t)(1+tx)}\mathrm{d}t\mathrm{d}x\right). \end{aligned}$$

If $\theta\in\left(\pi,\frac{3\pi}{2}\right)$, then $\cos\theta\leq 0$ and from Theorem 3 we get that

$$\int_0^1\int_0^1\frac{t^2(1-x)}{(1+t)(1+t^2-2t\cos\theta)}\mathrm{d}t\mathrm{d}x\leq 2\int_0^1\frac{t^2(1-t)}{(1+t)(1+t^2-2t\cos\theta)}\mathrm{d}t. \tag{11}$$

The inequality of Cauchy-Schwarz implies

$$\begin{aligned}(12)\qquad &\left(\int_0^1\!\!\int_0^1 \frac{t^2(1-x)}{(1+t)\sqrt{(1+tx)(1+t^2-2t\cos\theta)}}\mathrm{d}t\mathrm{d}x\right)^2\\ &\le \int_0^1\!\!\int_0^1 t^2\frac{1-x}{(1+t)(1+tx)}\mathrm{d}t\mathrm{d}x \int_0^1\!\!\int_0^1 \frac{t^2(1-x)}{(1+t)(1+t^2-2t\cos\theta)}\mathrm{d}t\mathrm{d}x.\end{aligned}$$

At the next step we prove that

$$\begin{aligned}(13)\qquad &\sin\frac{\theta}{2}\int_0^1\!\!\int_0^1 t^2\frac{(1-t^2x)(1-x)}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\mathrm{d}t\mathrm{d}x\\ &\le \int_0^1\!\!\int_0^1 \frac{t^2(1-x)}{(1+t)\sqrt{(1+tx)(1+t^2-2t\cos\theta)}}\mathrm{d}t\mathrm{d}x,\ \theta\in\left[\pi,\frac{17\pi}{12}\right].\end{aligned}$$

The above inequality is valid because

$$(14)\qquad \frac{\sin\frac{\theta}{2}}{1+t^2-2t\cos\theta}\le\frac{1}{(1+t)\sqrt{1+t^2-2t\cos\theta}},$$

which is equivalent to $0\le(1-t)^2(1+\cos\theta),\ \theta\in[\pi,\frac{17\pi}{12}],\ t\in[0,1]$ and

$$(15)\qquad \frac{1-t^2x}{1+t^2x^2-2tx\cos\theta}\le\frac{1}{\sqrt{1+tx}},\quad \theta\in\left[\pi,\frac{17\pi}{12}\right],\quad t\in[0,1].$$

For the argumentation of the inequality (15) we observe that $\theta\in[\pi,\frac{17\pi}{12}]$ implies $\cos\theta\le-\frac{1}{4}$ and so, instead of (15), it is enough to prove the following stronger inequality:

$$\frac{1-t^2x^2}{1+t^2x^2+\frac{1}{2}tx}\le\frac{1}{\sqrt{1+tx}},$$

which is equivalent to $t^5x^5\le\frac{17}{4}t^2x^2+3t^3x^3,\ t,x\in[0,1]$. By multiplying (14) and (15) it results that

$$\frac{t^2(1-t^2x)(1-x)\sin\frac{\theta}{2}}{(1+t^2-2t\cos\theta)(1+t^2x^2-2tx\cos\theta)}\le\frac{t^2(1-x)}{(1+t)\sqrt{(1+tx)(1+t^2-2t\cos\theta)}},$$

and integrating on the suitable domain we get (13).

Now (10), (11), (12), (13) imply that $\Delta(\theta)\le 0$ if $\theta\in[\pi,\frac{17\pi}{12}]$ and so $L_1(\theta,T)\ge 0$ for $\theta\in\left[\pi,\frac{17\pi}{12}\right]$, $T\in R$. According to (8) and (9) we have proved that $\theta\in\left(0,\frac{17\pi}{12}\right)$ implies $M(\theta,T)\ge\frac{1}{2}-\ln 2$, $T\in R$. It remains to discuss the case $\theta\in\left[\frac{17\pi}{12},2\pi\right)$. To prove in this case $M(\theta,T)\ge\frac{1}{2}-\ln 2$, $T\in R$, we will deduce on other integral representation of a Fourier series. Let $\Gamma=\gamma_1\cup\gamma_3\cup\gamma_2\cup\gamma_4$ denote the contour constructed by the following curves $\gamma_1(t)=R\mathrm{e}^{\mathrm{i}t},\gamma_2(t)=r\mathrm{e}^{-\mathrm{i}t}$, $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\gamma_3(t)=\mathrm{i}R+t(\mathrm{i}r-\mathrm{i}R),\gamma_4(t)=-\mathrm{i}r+t(\mathrm{i}r-\mathrm{i}R)$, $t\in[0,1]$. We calculate the integral $\int_\Gamma f(z)\mathrm{d}z$, where $f(z)=\frac{\mathrm{e}^{\mathrm{i}\theta z}}{(\beta+z)(\mathrm{e}^{2\pi\mathrm{i}z}-1)}$, $\beta>0$, using the residue theory. It is simple to deduce that $\lim_{R\to\infty}\int_{\gamma_1}f(z)\mathrm{d}z=$

0, $\mathrm{Rez}(f,k)=\frac{\mathrm{e}^{\mathrm{i}\theta k}}{2\pi\mathrm{i}(k+\beta)}$, $\lim_{r\to 0}\int_{\gamma_2} f(z)\mathrm{d}z=-\mathrm{i}\pi\cdot\mathrm{Rez}(f,0)$ and we get the equality

$$\lim_{\substack{R\to\infty\\ r\to 0}}\int_{\gamma_3\cup\gamma_4} f(z)\mathrm{d}z=\sum_{k=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\theta k}}{k+\beta}$$

or, equivalently, for $\theta\in(0,2\pi)$ and $\beta\in(0,\infty)$ the following identity holds:

$$\frac{1}{2\beta}+\sum_{n=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}n\theta}}{n+\beta}=\int_0^{\infty}\frac{x(\mathrm{e}^{(2\pi-\theta)x}+\mathrm{e}^{\theta x})}{(\beta^2+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x+\mathrm{i}\beta\int_0^{\infty}\frac{\mathrm{e}^{(2\pi-\theta)x}-\mathrm{e}^{\theta x}}{(\beta^2+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x. \tag{16}$$

Using the deduced identity, some calculations lead us to:

$$\begin{aligned} M(\theta,T) &= \frac{1}{2}-\ln 2+\frac{1}{1+T^2}\Bigg(\int_0^{\infty}\frac{x(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x}-\mathrm{e}^{\frac{1}{2}\theta x})^2}{(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x\\ &\quad+3T\int_0^{\infty}\frac{x^2(\mathrm{e}^{(2\pi-\theta)x}-\mathrm{e}^{\theta x})}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x\\ &\quad+3T^2\int_0^{\infty}\frac{x(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x}-\mathrm{e}^{\frac{1}{2}\theta x})^2}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x+T^2(3\ln 2-2)\Bigg). \end{aligned}$$

If we put $\theta=\pi$, $\beta=1$ and $\beta=2$ in (16) we obtain

$$\int_0^{\infty}\frac{3x\mathrm{e}^{\pi x}}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x=\ln 2-\frac{5}{8}$$

and so we get that

$$\frac{23}{20}\int_0^{\infty}\frac{3x\mathrm{e}^{\pi x}}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x\le 3\ln 2-2.$$

from which it follows the inequality

$$\begin{aligned} M(\theta,T) &\ge \frac{1}{2}-\ln 2+\frac{1}{1+T^2}\Bigg(\int_0^{\infty}\frac{x\left(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x}-\mathrm{e}^{\frac{1}{2}\theta x}\right)^2}{(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x\\ &\quad+3T\int_0^{\infty}\frac{x^2(\mathrm{e}^{(2\pi-\theta)x}-\mathrm{e}^{\theta x})}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x\\ &\quad+3T^2\int_0^{\infty}\frac{x\left(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x}-\mathrm{e}^{\frac{1}{2}\theta x}\right)^2+\frac{23}{20}x\mathrm{e}^{\pi x}}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x\Bigg),\quad \theta\in\left[\frac{17\pi}{12},2\pi\right]. \end{aligned} \tag{17}$$

Let introduce the notation

$$\begin{aligned} L_3(\theta,T) &= \int_0^{\infty}\frac{x\left(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x}-\mathrm{e}^{\frac{1}{2}\theta x}\right)^2}{(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x\\ &\quad+3T\int_0^{\infty}\frac{x^2(\mathrm{e}^{(2\pi-\theta)x}-\mathrm{e}^{\theta x})}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x \end{aligned}$$

$$+ 3T^2 \int_0^\infty \frac{x\left(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x} - \mathrm{e}^{\frac{1}{2}\theta x}\right)^2 + \frac{23}{20}x\mathrm{e}^{\pi x}}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x.$$

We prove that $L_3(\theta, T) \geq 0$, $\theta \in [\frac{17\pi}{12}, 2\pi)$, $T \in R$. The discriminant of L_3 is:

$$\begin{aligned}
\Delta_3(\theta) \quad &= \quad \left(3\int_0^\infty \frac{x^2(\mathrm{e}^{(2\pi-\theta)x} - \mathrm{e}^{\theta x})}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x\right)^2 \\
&\quad -12\int_0^\infty \frac{x\left(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x} - \mathrm{e}^{\frac{1}{2}\theta x}\right)^2}{(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x \\
&\quad \cdot \int_0^\infty \frac{x\left(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x} - \mathrm{e}^{\frac{1}{2}\theta x}\right)^2 + \frac{23}{20}x\mathrm{e}^{\pi x}}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x.
\end{aligned}$$

From the inequality of Cauchy-Schwarz it follows that

(18)

$$\begin{aligned}
&\left(\int_0^\infty \frac{x\left|\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x} - \mathrm{e}^{\frac{1}{2}\theta x}\right|\sqrt{(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x} - \mathrm{e}^{\frac{1}{2}\theta x})^2 + \frac{23}{20}\mathrm{e}^{\pi x}}}{\sqrt{1+x^2}(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x\right)^2 \\
&\leq \int_0^\infty \frac{x\left(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x} - \mathrm{e}^{\frac{1}{2}\theta x}\right)^2 + \frac{23}{20}x\mathrm{e}^{\pi x}}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x \int_0^\infty \frac{x\left(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x} - \mathrm{e}^{\frac{1}{2}\theta x}\right)^2}{(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x.
\end{aligned}$$

If we prove that

(19)

$$\begin{aligned}
&\left|\int_0^\infty \frac{x^2(\mathrm{e}^{(2\pi-\theta)x} - \mathrm{e}^{\theta x})}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x\right| = \int_0^\infty \frac{x^2\left|\mathrm{e}^{(2\pi-\theta)x} - \mathrm{e}^{\theta x}\right|}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x \\
&\qquad \leq \frac{2}{\sqrt{3}} \cdot \int_0^\infty \frac{x\left|\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x} - \mathrm{e}^{\frac{1}{2}\theta x}\right|\sqrt{(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x} - \mathrm{e}^{\frac{1}{2}\theta x})^2 + \frac{23}{20}x\mathrm{e}^{\pi x}}}{\sqrt{1+x^2}(4+x^2)(\mathrm{e}^{2\pi x}-1)}\mathrm{d}x,
\end{aligned}$$

then from (18) and (19) it is easy to deduce that $\Delta_3(\theta) \leq 0$ and so we get $L_3(\theta, T) \geq 0$, $\theta \in [\frac{17\pi}{12}, 2\pi)$, $T \in R$.

In order to prove (19) it is enough to show that for every $x \in [0, \infty)$, $\theta \in [\frac{17\pi}{12}, 2\pi)$, the next inequality holds true:

$$\begin{aligned}
&\frac{x^2\left|\mathrm{e}^{(2\pi-\theta)x} - \mathrm{e}^{\theta x}\right|}{(1+x^2)(4+x^2)(\mathrm{e}^{2\pi x}-1)} \\
&\leq \frac{2}{\sqrt{3}} \frac{x\left|\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x} - \mathrm{e}^{\frac{1}{2}\theta x}\right|\sqrt{(\mathrm{e}^{\frac{1}{2}(2\pi-\theta)x} - \mathrm{e}^{\frac{1}{2}\theta x})^2 + \frac{23}{20}\mathrm{e}^{\pi x}}}{\sqrt{1+x^2}(4+x^2)(\mathrm{e}^{2\pi x}-1)}.
\end{aligned}$$

The last inequality is equivalent to

$$\frac{x}{\sqrt{1+x^2}} \leq \frac{2}{\sqrt{3}}\sqrt{\frac{\mathrm{e}^{2(\theta-\pi)x}-0,85\mathrm{e}^{(\theta-\pi)x}+1}{(\mathrm{e}^{(\theta-\pi)x}+1)^2}}, \tag{20}$$

where $\theta \in \left[\frac{17\pi}{12}, 2\pi\right]$, $x \in [0,\infty)$. It is enough to prove the inequality (20) for $\theta = \frac{17\pi}{12}$. Let $g, h : [0,\infty) \to R$ be two functions defined by the equalities

$$g(x) = \frac{2}{\sqrt{3}}\sqrt{\frac{\mathrm{e}^{\frac{5\pi}{6}x}-0.85\mathrm{e}^{\frac{5\pi}{12}x}+1}{(\mathrm{e}^{\frac{5\pi}{12}x}+1)^2}}\ , \quad h(x) = \frac{x}{\sqrt{1+x^2}}\ .$$

Because the functions g and h are increasing and $g(0) > h\left(\frac{1}{2}\right)$, $g\left(\frac{1}{2}\right) > h\left(\frac{3}{4}\right)$, $g\left(\frac{3}{4}\right) > h(1)$, $g(1) > h\left(\frac{3}{2}\right)$, $g\left(\frac{3}{2}\right) > h(2)$, it results the inequality $g(x) > h(x)$, $x \in [0,2]$, and we observe that $g(x) > 1 > h(x)$ for $x \in [2,\infty)$, so it follows that $g(x) > h(x)$ for every $x \in [0,\infty)$.

From (18) and (19) we get $\Delta_3(\theta) \leq 0$, $\theta \in \left[\frac{17\pi}{12}, 2\pi\right)$, $L_3(\theta,T) \geq 0$, $\theta \in \left[\frac{17\pi}{12}, 2\pi\right)$, $T \in \mathbb{R}$ and, according to (17), we have $M(\theta,T) \geq \frac{1}{2} - \ln 2$, $\theta \in \left[\frac{17\pi}{12}, 2\pi\right)$, $T \in \mathbb{R}$. In conclusion, $M(\theta,T) \geq \frac{1}{2} - \ln 2$, $\theta \in (0,2\pi)$, $T \in R$ and $M(\pi,0) = \frac{1}{2} - \ln 2$, which means that $\min\{M(\theta,T) \mid \theta \in (0,2\pi), T \in R\} = \frac{1}{2} - \ln 2$.

The necessary and sufficient condition for starlikeness (5) is valid if and only if $m = \frac{1}{4(1+c)} + \frac{1}{2} - \ln 2 \geq 0$ (m is defined by (6)), which is equivalent to $c = \frac{3-4\ln 2}{4\ln 2 - 2}$. □

Theorem 5. *The value* $c = \frac{3-4\ln 2}{4\ln 2-2}$ *is the biggest having the property that the condition*

$$f \in A, \quad \mathrm{Re}\left(f'(z) + \frac{z}{2}f''(z)\right) > -c, \quad z \in U$$

implies the univalence of the function f.

Proof. It is well-known that a necessary condition for the univalence of a function $f \in A$ is $f'(z) \neq 0$, $z \in U$. We have proved at the previous theorem that the condition (1) implies

$$f(z) = z + 4(1+c)\sum_{n=2}^{\infty}\frac{z^n}{n(n+1)}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}(n-1)t}\mathrm{d}\mu(t), \quad z \in U.$$

From this integral representation it results that

$$\begin{aligned} f'(z) &= 1 + 4(1+c)\sum_{n=2}^{\infty}\frac{z^{n-1}}{n+1}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}(n-1)t}\mathrm{d}\mu(t) = \\ &= \left(1 + 2\sum_{n=1}^{\infty} z^n\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}\mu(t)\right) * \left(1 + 2(1+c)\sum_{n=1}^{\infty}\frac{z^n}{n+2}\right). \end{aligned}$$

Theorem 1 (The duality principle) implies that the condition $f'(z) \neq 0$, $z \in U$ holds if and only if

$$\mathrm{Re}\left(1 + 2(1+c)\sum_{n=1}^{\infty} \frac{z^n}{n+2}\right) > \frac{1}{2}, \quad z \in U. \tag{21}$$

Because

$$\inf_{z \in U} \mathrm{Re}\left(1 + 2(1+c)\sum_{n=1}^{\infty} \frac{z^n}{n+2}\right) = 1 + 2(1+c)\left(\frac{1}{2} - \ln 2\right),$$

we get that condition (21) is equivalent to $1 + 2(1+c)\left(\frac{1}{2} - \ln 2\right) \geq \frac{1}{2}$ and $c \leq \frac{3-4\ln 2}{4\ln 2 - 2}$. □

4. INTEGRAL VERSION OF THE RESULT

Let $L : A \to A$ be the operator of Libera defined by $L(f)(z) = \frac{1}{z}\int_0^z f(t)\mathrm{d}t$. For a given real number c, we introduce the following subclass of analytic functions: $R_c = \{f \in A | \mathrm{Re} f'(z) > -c, z \in U\}$.

Theorem 6. *The biggest value of c for which $L(f) \in S^*$ for every $f \in R_c$ is $c = \frac{3-4\ln 2}{4\ln 2 - 2}$. For a bigger value of c than $\frac{3-4\ln 2}{4\ln 2 - 2}$, there exists $f \in R(c)$ such that $L(f)$ is not even univalent in U.*

Proof. By derivation from $F(z) = \frac{1}{z}\int_0^z f(t)\mathrm{d}t$ we get the equality $F'(z) + \frac{z}{2}F''(z) = f'(z)$, $z \in U$ and the function F satisfies the condition (1). Theorems 4 and 5 imply that the assertions of Theorem 6 are valid. □

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Department of Mathematics
Sapientia University
Piaţa Trandafirilor 61
540053 Târgu Mureş, Romania
E-mail: szasz_robert2001@yahoo.com