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EXPRESSING THE CYCLIC MODULES IN TERMS OF ELEMENTARY MODULES IN THE CLASSICAL HALL ALGEBRA

CSABA SZÁNTÓ

Abstract. Using some results obtained in the Hall algebra of the Kronecker algebra we obtain a new recursive algorithm for expressing the indecomposable (cyclic) modules in terms of semisimple (elementary) modules in the classical Hall algebra.

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1. PRELIMINARIES

Let $K\ :\ 1 \overset{\alpha}{\underset{\beta}{\leftleftarrows}} 2$ be the Kronecker-quiver, k a finite field with $|k| = q_k$ and kK the corresponding path-algebra over k, called Kronecker algebra. We will consider the category mod-kK of finitely generated (hence finite) right modules over kK, which will be identified with the category rep-kK of the finite dimensional k-representations of the Kronecker quiver. For general notions concerning the representation theory of quivers, we refer to [1] or [2].

Up to isomorphism we will have two simple objects in mod-kK corresponding to the two vertices. We shall denote them by S_1 and S_2. For a module $M \in$ mod-kK, $[M]$ will denote the isomorphism class of M. For a module M let $tM := M \oplus ... \oplus M$ (t-times).

The indecomposable modules in mod-kK are divided into three families: the preprojectives, the regulars and the preinjectives.

The preprojective indecomposables (seen as representations) up to isomorphism have the following form:

$$P_n:\ k^{n+1} \overset{\begin{pmatrix} I \\ 0 \end{pmatrix}}{\underset{\begin{pmatrix} 0 \\ I \end{pmatrix}}{\leftleftarrows}} k^n \text{ , where } n \in \mathbb{N}.$$

The preinjective indecomposables are isomorphic to:

$$I_n:\ k^n \overset{(0I)}{\underset{(I0)}{\leftleftarrows}} k^{n+1} \text{ , where } n \in \mathbb{N}.$$

We mention here that up to isomorphism $P_0 = S_1, P_1$ are the projective and $I_0 = S_2, I_1$ are the injective indecomposables.

Viewed as finite dimensional k-representations of the Kronecker quiver, the regular indecomposables up to isomorphism are:

$$R_1^o(t) := \; k[X]/(X^t) \underset{id}{\overset{X}{\leftleftarrows}} k[X]/(X^t) \; ,$$

$$R_1^\mu(t) := \; k[X]/((X-\mu)^t) \underset{X}{\overset{id}{\leftleftarrows}} k[X]/((X-\mu)^t) \; ,$$

where $t \geq 1$ and $\mu \in k$;

$$R_l^{\varphi_l}(t) := \; k[X]/(\varphi_l(X)^t) \underset{X}{\overset{id}{\leftleftarrows}} k[X]/(\varphi_l(X)^t) \; ,$$

where $t \geq 1$, $l \geq 2$ and $\varphi_l(X)$ is a monic irreducible polynomial of degree l over k.

Let $N(q_k, l) = \frac{1}{l}\sum_{d|l} \mu(\frac{l}{d}) q_k^d$, where $l \geq 1$, and μ is the Möbius function. It is well known that $N(q_k, l)$ is the number of monic, irreducible polynomials of degree l over a field with q elements.

Let $M(q_k, l) := N(q_k, l)$ when $l \geq 2$ and $M(q_k, 1) := N(q_k, 1) + 1 = q_k + 1$.

To somewhat simplify the notations, we shall fix in an arbitrary way bijections $f_1 : \{\mu | \mu \in k\} \cup \{o\} \to \{1, ..., q+1\}$ and $f_l : \{\varphi_l | \varphi_l$ monic irreducible polynomial of degree l over k $\} \to \{1, ..., N(q_k, l)\}$ (where $l \geq 2$) and then let $R_1^o(t) = R_1^{f_1(o)}(t)$, $R_1^\mu(t) = R_1^{f_1(\mu)}(t)$, $R_l^{\varphi_l}(t) = R_l^{f_l(\varphi_l)}(t)$. So, using the notations above, our regular indecomposables are $R_l^a(t)$, where $l \geq 1$, $a = \overline{1, M(q_k, l)}$, $t \geq 1$.

Using the terminology of the Auslander-Reiten theory (see [1] or [2]) we say that a sequence of the form $[R_l^a(1)], ..., [R_l^a(t)], ...$ is the vertex-sequence of a homogeneous tube T_l^a. In this terminology, the regular indecomposable $R_l^a(1)$ is called quasi-simple and $R_l^a(t)$ is of quasi-length t and quasi-socle $R_l^a(1)$. A module with all its indecomposable direct summands in the tube T_l^a will be denoted by R_l^a.

The Hall algebra $\mathcal{H}(kK)$ associated to the Kronecker algebra kK is the $\mathbb{Q}$-space having as basis the isomorphism classes in mod-kK together with a multiplication (the so-called Hall product) defined by:

$$[N_1][N_2] = \sum_{[M]} F_{N_1 N_2}^M [M].$$

The structure constants $F_{N_1 N_2}^M = |\{M \supseteq U | \; U \cong N_2, \; M/U \cong N_1\}|$ are called Hall numbers. It is easy to see that the Hall-algebra is a well-defined, associative, usually noncommutative algebra with unit element the isomorphism class of the zero module.

We will fix now a homogeneous tube $T := T_l^a$ with indecomposable regulars $R(t) := R_l^a(t)$ and modules $R := R_l^a$. Let $q := q_k^l$. The isomorphism classes $[R]$ (and $[0]$) form a $\mathbb{Q}$-basis of a unital $\mathbb{Q}$-subalgebra $\mathcal{H}(T)$ of $\mathcal{H}(kK)$, called the Hall algebra of the tube T.

We know that $\mathcal{H}(T)$ coincides with the classical Hall algebra studied by Ph. Hall (see [4]), moreover, the indecomposable $R(t)$ corresponds in classical terms to a cyclic module and the quasi-semisimple $tR(1)$ to a so called elementary module. We will use the notation $u_{(t)} := [R(t)]$ and $u_{(1^t)} := [tR(1)]$ for the isoclasses of cyclic, respectively elementary modules.

The following theorem claims that the classical Hall algebra is a polynomial algebra over $\mathbb{Q}$ with generators the elementary (respectively the cyclic) modules. More precisely:

Theorem 1. ([4]) *We have* $\mathcal{H}(T) = \mathbb{Q}[u_{(1)}, u_{(1^2)}, ..., u_{(1^t)}, ...]$ *and* $\mathcal{H}(T) = \mathbb{Q}[u_{(1)}, u_{(2)}, ..., u_{(t)}, ...]$.

It follows that

$$(*) \qquad u_{(n)} = \sum_{\lambda=(\lambda_1,...,\lambda_s)\in\mathbb{P}(n)} b_\lambda u_{(1^{\lambda_1})}...u_{(1^{\lambda_s})},$$

where $b_\lambda \in \mathbb{Q}$ and $\mathbb{P}(n)$ is the set of partitions of n. We already know that $b_{(1^n)} = 1$. Our aim is to give a recursive algorithm for computing the other coefficients b_λ, using some formulas obtained by the author in the Hall algebra of the Kronecker algebra.

2. THE RECURSIVE ALGORITHM

We start from some formulas obtained by the author in [3]:

$$(1) \qquad u_{(1^t)}[P_m] = q^t[P_m]u_{(1^t)} + [P_{m+l}]u_{(1^{t-1})},$$

$$(2) \qquad u_{(t)}[P_m] = q^t[P_m]u_{(t)} + [P_{m+lt}] + \sum_{i=1}^{t-1}(q^{(t-i)} - q^{(t-i-1)})[P_{m+li}]u_{(t-i)}.$$

Let $\lambda = (\lambda_1, ..., \lambda_s) \in \mathbb{P}(n)$, such that $\lambda_1 = ... = \lambda_t > \lambda_{t+1}$ and denote $\bar{\lambda} := (\lambda_1 - 1, ..., \lambda_t - 1, \lambda_{t+1}, ..., \lambda_s)$. We will then compute $u_{(n)}[S_1]$ in two ways. Firstly, we apply (*) and then formula (1). Secondly, we apply formula (2) and then (*). Comparing the coefficients of $[P_{lt}]u_{(1^{\lambda_1-1})}...u_{(1^{\lambda_t-1})}u_{(1^{\lambda_{t+1}})}...u_{(1^{\lambda_s})}$ in the two cases, we get the formula

$$q^{n-t\lambda_1}b_\lambda + \sum_{\mu\in\mathbb{P}(n)\setminus\{\lambda\},\bar{\lambda}\dashv\mu} c(\mu,\lambda)q^{n-\sum_{i,\bar{\lambda}_i=\mu_i-1}\mu_i}b_\mu = (q^{n-t} - q^{n-t-1})b_{\bar{\lambda}},$$

where $c(\mu, \lambda)$ is the number of compositions β, such that $\beta \sim \bar{\lambda}$ and $\beta \dashv \mu$. Here a composition $\beta = (\beta_1, \beta_2, \ldots)$ is a sequence of non-negative integers with only a finite number of non-zero terms. So, a partition is a composition with

$\beta_1 \geq \beta_2 \geq \ldots$. Two compositions α, β are conjugate (in notation $\alpha \sim \beta$) if they give the same partition after reordering if it is necessary. We write $\beta \dashv \alpha$ if we have $\alpha_i - 1 \leq \beta_i \leq \alpha_i$, $\forall i \geq 1$.

Using the recursive formula above, we can recursively compute the coefficients b_λ. We present here the first steps of the new algorithm:

For $n = 1$ we get

$$b_{(1)} = 1.$$

For $n = 2$ we get

$$b_{(11)} = 1$$
$$b_{(2)} + 2qb_{(11)} = (q-1)b_{(1)}$$

so $b_{(2)} = -(q+1)$.

For $n = 3$ we get

$$b_{(111)} = 1$$
$$qb_{(21)} + 3q^2 b_{(111)} = (q^2 - q)b_{(11)}$$
$$b_{(3)} + q^2 b_{(21)} = (q^2 - q)b_{(2)}$$

so $b_{(21)} = -(2q+1)$, $b_{(3)} = q^3 + q^2 + q$.

For $n = 4$ we get

$$b_{(1111)} = 1$$
$$q^2 b_{(211)} + 4q^3 b_{(1111)} = (q^3 - q^2)b_{(111)}$$
$$b_{(22)} + 2qb_{(211)} + C_4^2 q^2 b_{(1111)} = (q^2 - q)b_{(11)}$$
$$qb_{(31)} + 2q^2 b_{(22)} + 2q^3 b_{(211)} = (q^3 - q^2)b_{(21)}$$
$$b_{(4)} + q^3 b_{(31)} = (q^3 - q^2)b_{(3)}$$

so $b_{(211)} = -(3q+1)$, $b_{(22)} = q^2 + q$, $b_{(31)} = 2q^3 + q^2 + q$ and $b_{(4)} = -(q^6 + q^5 + q^4 + q^3)$.

We can continue in the same way.

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Faculty of Mathematics and Computer Science
"Babeş-Bolyai" University
Str. M. Kogălniceanu nr. 1
400084 Cluj-Napoca, Romania
E-mail: szanto@math.ubbcluj.ro