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(L^p, L^q)-COMPLETE ADMISSIBILITY AND EXPONENTIAL EXPANSIVENESS OF LINEAR SKEW-PRODUCT FLOWS

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Abstract. The goal of this paper is to give necessary and sufficient conditions for uniform exponential expansiveness of time-varying systems modelled by linear skew-product flows in infinite-dimensional spaces. If $p, q \in [1, \infty)$ we prove that the complete admissibility of the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is a sufficient condition for uniform exponential expansiveness and it becomes necessary if and only if $p \leq q$.

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1. INTRODUCTION

The input-output characterizations play a very important role in the study of the asymptotic properties of evolution equations. An impressive list of interesting problems concerning the connections between stability and control were solved by means of the input-output conditions (see [1], [3], [4], [6], [7], [8], [12], [14], [16]). In recent years the asymptotic behavior of linear skew-product flows was at the center of intensive studies (see [3], [4], [6], [7], [8], [12], [13], [14], [15]). Exponential stability was characterized in [8], exponential expansiveness was analyzed in [7] and new concepts of exponential dichotomy were introduced and studied in [3], [4] and [6].

The main idea in [7] was to associate to a linear skew-product flow $\pi = (\Phi, \sigma)$, on $X \times \Theta$, the integral equation (E_c^{θ,t_0}), for every $(\theta, t_0) \in \Theta \times \mathbb{R}_+$, where

$$f_{\theta,u}(t) = \Phi(\sigma(\theta, s), t - s)f_{\theta,u}(s) + \int_s^t \Phi(\sigma(\theta, \tau), t - \tau)u(\tau)\,\mathrm{d}\tau, \quad \forall t \geq s \geq t_0$$

with $f, u \in C_0([t_0, \infty), X)$. One of the main results in [7] gives a characterization for uniform exponential expansiveness of π in terms of the unique solvability of the equations (E_c^{θ,t_0}), for every $(\theta, t_0) \in \Theta \times \mathbb{R}_+$.

Naturally, the question arises whether uniform exponential expansiveness of linear skew-product flows can be characterized in terms of L^p-spaces.

The central purpose of this paper is to answer this question. In what follows we present a new and unified approach for the uniform exponential expansiveness of linear skew-product flows, our arguments being based on input-output techniques. We present a detailed and constructive study for the uniform exponential expansiveness of linear skew-product flows by means of the solvability of an integral equation. We consider a special input space $L^q(\mathbb{R}_+, X)$ and we analyze the connections between the uniform exponential expansiveness and the complete admissibility of the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$, with $p, q \in [1, \infty)$. We prove that the uniform complete admissibility of the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ implies the uniform exponential expansiveness. An example will motivate our hypothesis and methods. Finally, for $p \leq q$ we obtain that a linear skew-product flow $\pi = (\Phi, \sigma)$ is uniformly exponentially expansive if and only if the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is uniformly completely admissible for it.

2. EXPONENTIAL EXPANSIVENESS OF LINEAR SKEW-PRODUCT FLOWS

Let X be a real or a complex Banach space, let (Θ, d) be a metric space and let $\mathcal{E} = X \times \Theta$. The norm on X and on $\mathcal{L}(X)$-the Banach algebra of all bounded linear operators on X will be denoted by $||\cdot||$. Denote by I the identity operator on X.

Definition 1. A mapping $\sigma : \Theta \times \mathbb{R} \to \Theta$ is called a *flow* on Θ if $\sigma(\theta, 0) = \theta$, for all $\theta \in \Theta$ and $\sigma(\theta, t + s) = \sigma(\sigma(\theta, t), s)$, for all $(\theta, t, s) \in \Theta \times \mathbb{R}^2$.

Definition 2. A pair $\pi = (\Phi, \sigma)$ is called *linear skew-product flow* on $\mathcal{E}$ if σ is a flow on Θ and $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$ has the following properties:

(i) $\Phi(\theta, 0) = I$, for all $\theta \in \Theta$;

(ii) $\Phi(\theta, t + s) = \Phi(\sigma(\theta, s), t)\Phi(\theta, s)$ (*the cocycle identity*), for all $(\theta, t, s) \in \Theta \times \mathbb{R}_+^2$;

(iii) there are $M, \omega > 0$ such that $||\Phi(\theta, t)|| \leq M\mathrm{e}^{\omega t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(iv) for every $x \in X$ the mapping $(\theta, t) \mapsto \Phi(\theta, t)x$ is continuous.

For interesting examples of linear skew-product flows we refer to [3], [4], [12], [13].

Definition 3. A linear skew-product flow $\pi = (\Phi, \sigma)$ is said to be *uniformly exponentially expansive* if for every $(\theta, t) \in \Theta \times \mathbb{R}_+$ the operator $\Phi(\theta, t)$ is invertible and there are two constants $K, \nu > 0$ such that

$$||\Phi(\theta, t)x|| \geq K\mathrm{e}^{\nu t}||x||, \quad \forall (x, \theta, t) \in \mathcal{E} \times \mathbb{R}_+.$$

For every $p \in [1, \infty)$ consider $L^p(\mathbb{R}_+, X)$ the space of all Bochner measurable functions $f : \mathbb{R}_+ \to X$ with $\int_0^\infty ||f(\tau)||^p \, \mathrm{d}\tau < \infty$ which is a Banach space with respect to the norm

$$||f||_p := \left(\int_0^\infty ||f(\tau)||^p \, \mathrm{d}\tau \right)^{1/p}.$$

Definition 4. Let $p, q \in [1, \infty)$. The pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is said to be *completely admissible* for $\pi = (\Phi, \sigma)$ if for every $\theta \in \Theta$ and every $v \in$

$L^q(\mathbb{R}_+, X)$ there is a unique continuous function $f \in L^p(\mathbb{R}_+, X)$ such that the pair (f, v) verifies the integral equation

$$(E_\theta)\ \ f(t) = \Phi(\sigma(\theta, s), t-s)f(s) + \int_s^t \Phi(\sigma(\theta,\tau), t-\tau)v(\tau)\ \mathrm{d}\tau, \quad \forall t \geq s \geq 0.$$

THEOREM 1. *If the pair* $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ *is completely admissible for* $\pi = (\Phi, \sigma)$*, then* $\Phi(\theta, t)$ *is invertible, for all* $(\theta, t) \in \Theta \times \mathbb{R}_+$.

Proof. Let $(\theta, t_0) \in \Theta \times \mathbb{R}_+^*$ and let $x \in \operatorname{Ker} \Phi(\theta, t_0)$. We consider the functions

$$f_1, f_2 : \mathbb{R}_+ \to X, \quad f_1(t) = \Phi(\theta, t)x, \quad f_2(t) = 0.$$

We have that $f_1, f_2 \in L^p(\mathbb{R}_+, X)$ and an easy computation shows that the pairs $(f_1, 0)$ and $(f_2, 0)$ verify the equation (E_θ). It follows that $f_1 = f_2$ which shows that $x = f_1(0) = 0$. So $\Phi(\theta, t_0)$ is injective.

Let now $y \in X$. We consider a continuous function $\alpha : \mathbb{R}_+ \to [0,1]$ with compact support contained in (t_0, t_0+2) and with $\int_{t_0}^\infty \alpha(\tau)\ \mathrm{d}\tau = 1$. We define the function

$$v : \mathbb{R}_+ \to X \quad v(t) = \begin{cases} -\alpha(t)\Phi(\sigma(\theta, t_0), t-t_0)y, & t \geq t_0 \\ 0 & , \ t \in [0, t_0]. \end{cases}$$

We have that $v \in L^q(\mathbb{R}_+, X)$ and then from hypothesis there is $f \in L^p(\mathbb{R}_+, X)$ such that the pair (f, v) satisfies the equation (E_θ). In particular, it follows that

$$(1) \qquad f(t) = \Phi(\sigma(\theta, s), t-s)f(s) - \left(\int_s^t \alpha(\tau)\ \mathrm{d}\tau\right)\Phi(\sigma(\theta, t_0), t-t_0)y,$$

for all $t \geq s \geq t_0$. Let

$$g : [t_0, \infty) \to X, \quad g(t) = \int_t^\infty \alpha(\tau)\ \mathrm{d}\tau\ \Phi(\sigma(\theta, t_0), t-t_0)y.$$

We observe that

$$(2) \qquad g(t) = \Phi(\sigma(\theta, s), t-s)g(s) - \left(\int_s^t \alpha(\tau)\ \mathrm{d}\tau\right)\ \Phi(\sigma(\theta, t_0), t-t_0)y,$$

for all $t \geq s \geq t_0$. Setting

$$\varphi : \mathbb{R}_+ \to X, \quad \varphi(t) = f(t+t_0) - g(t+t_0),$$

we have that $\varphi \in L^p(\mathbb{R}_+, X)$. From relations (1) and (2) we deduce that

$$\varphi(t) = \Phi(\sigma(\theta, t_0+s), t-s)\varphi(s), \quad \forall t \geq s \geq 0,$$

so the pair $(\varphi, 0)$ verifies the equation $(E_{\sigma(\theta, t_0)})$. From hypothesis it follows that $\varphi = 0$. In particular, we obtain that $f(t_0) = g(t_0) = y$. Since $f(t_0) = \Phi(\theta, t_0)f(0)$ we have that $y \in \operatorname{Range} \Phi(\theta, t_0)$, so $\Phi(\theta, t_0)$ is surjective. $\square$

REMARK 1. If the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is completely admissible for π, then for every $\theta \in \Theta$ it makes sense to consider the linear operator

$$\Gamma_\theta : L^q(\mathbb{R}_+, X) \to L^p(\mathbb{R}_+, X), \quad \Gamma_\theta v = f,$$

where $f \in L^p(\mathbb{R}_+, X)$ is continuous and the pair (f, v) verifies the equation (E_θ). It is easy to see that Γ_θ is a closed linear operator, so it is bounded.

DEFINITION 5. The pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is said to be *uniformly completely admissible* for π if it is completely admissible for π and there is $L > 0$ such that

$$||\Gamma_\theta|| \leq L, \quad \forall \theta \in \Theta.$$

In what follows for every set $A \subset \mathbb{R}$ we denote by χ_A its characteristic function.

The first main result of this paper is:

THEOREM 2. *If the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is uniformly completely admissible for π, then π is uniformly exponentially expansive.*

Proof. Step 1. Let $L > 0$ be given by Definition 5. We prove that there is $\alpha > 0$ such that

$$||\Phi(\theta, t)x|| \geq \alpha ||x||, \quad \forall (x, \theta, t) \in \mathcal{E} \times \mathbb{R}_+.$$

Let $(\theta, t_0) \in \Theta \times \mathbb{R}_+$ and let $x \in X$. We consider the functions

$$v : \mathbb{R}_+ \to X, \quad v(t) = -\chi_{[t_0+1, t_0+2)}(t)\Phi(\theta, t)x,$$

$$f : \mathbb{R}_+ \to X, \quad f(t) = \int_t^\infty \chi_{[t_0+1, t_0+2)}(\tau) \, \mathrm{d}\tau \; \Phi(\theta, t)x.$$

We have that $(f, v) \in L^p(\mathbb{R}_+, X) \times L^q(\mathbb{R}_+, X)$. An easy computation shows that the pair (f, v) verifies the equation (E_θ), so $f = \Gamma_\theta v$. It follows that

$$||f||_p \leq L||v||_q. \tag{3}$$

If $M, \omega > 0$ are given by Definition 2, then

$$||v(t)|| = \chi_{[t_0+1, t_0+2)}(t) \; ||\Phi(\theta, t)x|| \leq$$

$$\leq M\mathrm{e}^\omega ||\Phi(\theta, t_0 + 1)x|| \; \chi_{[t_0+1, t_0+2)}(t), \quad \forall t \geq 0,$$

so that

$$||v||_q \leq M\mathrm{e}^\omega ||\Phi(\theta, t_0 + 1)x||. \tag{4}$$

We observe that $f(t) = \Phi(\theta, t)x$, for all $t \in [0, 1]$. From

$$||\Phi(\theta, 1)x|| \; \chi_{[0,1)}(t) \leq M\mathrm{e}^\omega ||\Phi(\theta, t)x|| \; \chi_{[0,1)}(t) =$$

$$= M\mathrm{e}^\omega ||f(t)|| \; \chi_{[0,1)}(t) \leq M\mathrm{e}^\omega ||f(t)||, \quad \forall t \geq 0$$

we deduce that

$$||\Phi(\theta, 1)x|| \leq M\mathrm{e}^\omega ||f||_p. \tag{5}$$

From relations (3)–(5) it results that

$$\alpha\ ||\Phi(\theta,1)x|| \leq ||\Phi(\theta,t_0+1)x||, \tag{6}$$

where $\alpha = 1/LM^2\mathrm{e}^{2\omega}$. Since α does not depend on θ, t_0 or x we have that (6) holds for every $(x,\theta)\in\mathcal{E}$ and every $t_0\geq 0$. Taking into account that $\Phi(\theta,1)$ is surjective, it follows that

$$\alpha\ ||y|| \leq ||\Phi(\sigma(\theta,1),t_0)y||, \quad \forall (y,\theta)\in\mathcal{E}, \quad \forall t_0\geq 0. \tag{7}$$

Observing that $\sigma(\sigma(\theta,-1),1)=\theta$, for all $\theta\in\Theta$ we deduce that $\sigma_| : \Theta\times\{1\}\to\Theta$ is surjective. Hence we may write the relation (7) as follows

$$||\Phi(\theta,t_0)y|| \geq \alpha||y||, \quad \forall (y,\theta)\in\mathcal{E}, \quad \forall t_0\geq 0. \tag{8}$$

Step 2. Let $h>0$ be such that $h>(L\mathrm{e}/\alpha^2)^p$. Let $\theta\in\Theta$ and let $x\in X\setminus\{0\}$. We consider the functions

$$u:\mathbb{R}_+\to X, \quad u(t) = -\chi_{[h,2h)}(t)\ \frac{\Phi(\theta,t)x}{||\Phi(\theta,t)x||},$$

$$\varphi:\mathbb{R}_+\to X, \quad \varphi(t) = \int_t^\infty \frac{\chi_{[h,2h)}(s)}{||\Phi(\theta,s)x||}\ \mathrm{d}s\ \Phi(\theta,t)x.$$

We have that $(\varphi,u)\in L^p(\mathbb{R}_+,X)\times L^q(\mathbb{R}_+,X)$ and an easy computation shows that the pair (φ,u) satisfies the equation (E_θ). Then we have that

$$||\varphi||_p \leq L||u||_q = Lh^{1/q} \leq Lh. \tag{9}$$

Setting

$$a = \int_h^{2h} \frac{1}{||\Phi(\theta,s)x||}\ \mathrm{d}s,$$

we observe that $\varphi(t) = a\ \Phi(\theta,t)x$, for all $t\in[0,h)$. Then using relation (8) we obtain that

$$\alpha\ ||x||\ \chi_{[0,h)}(t) \leq ||\Phi(\theta,t)x||\ \chi_{[0,h)}(t) =$$
$$= \frac{||\varphi(t)||}{a}\ \chi_{[0,h)}(t)\ \leq \frac{||\varphi(t)||}{a}, \quad \forall t\geq 0,$$

which implies that

$$\alpha\ ||x||\ h^{1/p}\ \leq \frac{1}{a}\ ||\varphi||_p. \tag{10}$$

From relation (8) we have that

$$||\Phi(\theta,2h)x|| \geq \alpha\ ||\Phi(\theta,s)x||, \quad \forall s\in[h,2h),$$

which shows that

$$a \geq \frac{\alpha h}{||\Phi(\theta,2h)x||}. \tag{11}$$

From relations (10) and (11) we deduce that

$$\alpha^2 h^{(p+1)/p}||x|| \leq ||\Phi(\theta,2h)x||\ ||\varphi||_p. \tag{12}$$

Using relations (9) and (12) it follows that

$$\alpha^2 h^{(p+1)/p}||x|| \leq Lh\ ||\Phi(\theta, 2h)x||,$$

so that

$$||\Phi(\theta, 2h)x|| \geq (\alpha^2/L)h^{1/p}||x|| \geq \mathrm{e}\ ||x||. \tag{13}$$

Taking into account that h does not depend on θ or x we have that relation (13) holds for all $(x, \theta) \in \mathcal{E}$. Setting $\nu = 1/2h$ and $K = \alpha/\mathrm{e}$ we obtain that

$$||\Phi(\theta, t)x|| \geq K\mathrm{e}^{\nu t}||x||, \quad \forall (x, \theta, t) \in \mathcal{E} \times \mathbb{R}_+. \tag{14}$$

From Theorem 1 and relation (14) we deduce that π is uniformly exponentially expansive. □

Lemma 1. *Let $p, q \in [1, \infty)$ with $p \geq q$ and let $\nu > 0$. Then:*
(i) *for every $v \in L^q(\mathbb{R}_+, \mathbb{R}_+)$ the function*

$$f_v : \mathbb{R}_+ \to \mathbb{R}_+, \quad f_v(t) = \int_t^\infty \mathrm{e}^{-\nu(\tau - t)} v(\tau)\ \mathrm{d}\tau$$

belongs to $L^p(\mathbb{R}_+, \mathbb{R}_+)$;
(ii) *there is $\lambda = \lambda(p, q, \nu) > 0$ such that*

$$||f_v||_p \leq \lambda ||v||_q, \quad \forall v \in L^q(\mathbb{R}_+, \mathbb{R}_+).$$

Proof. It results using Hölder's inequality. □

The second main result of this paper is:

Theorem 3. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$ and let $p, q \in [1, \infty)$ with $p \geq q$. Then π is uniformly exponentially expansive if and only if the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is uniformly completely admissible for π.*

Proof. Necessity. Let $\theta \in \Theta$ and let $v \in L^q(\mathbb{R}_+, X)$. We consider the function

$$f : \mathbb{R}_+ \to X, \quad f(t) = -\int_t^\infty \Phi(\sigma(\theta, t), \tau - t)^{-1} v(\tau)\ \mathrm{d}\tau.$$

Then f is continuous and the pair (f, v) verifies the equation (E_θ). From Lemma 1 we also have that $f \in L^p(\mathbb{R}_+, X)$.

Let $\tilde{f} \in L^p(\mathbb{R}_+, X)$ be such that the pair $(\tilde{f}, v)$ satisfies the equation (E_θ). Setting $\varphi = f - \tilde{f}$ it follows that

$$\varphi(t) = \Phi(\theta, t)\varphi(0), \quad \forall t \geq 0. \tag{15}$$

If $K, \nu > 0$ are given by Definition 3 we have that

$$||\varphi(t)|| \geq K\mathrm{e}^{\nu t}||\varphi(0)||, \quad \forall t \geq 0. \tag{16}$$

Since $\varphi \in L^p(\mathbb{R}_+, X)$, from relation (16) it results that $\varphi(0) = 0$. Then from relation (15) we obtain that $\varphi \equiv 0$, so $\tilde{f} = f$. It follows that the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is completely admissible for π.

Let $\lambda = \lambda(p, q, \nu) > 0$ be given by Lemma 1. Then for every $\theta \in \Theta$ and every $v \in L^q(\mathbb{R}_+, X)$ we have that

$$||\Gamma_\theta v||_p \le \frac{\lambda}{K}||v||_q,$$

so the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is uniformly completely admissible for π.

Sufficiency. It results from Theorem 2. □

Remark 2. Generally, if $p < q$ and the linear skew-product flow π on $\mathcal{E} = X \times \Theta$ is uniformly exponentially expansive, it does not result that the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is uniformly completely admissible for π.

Example 1. Let $X = \Theta = \mathbb{R}$ and let $\sigma(\theta, t) = \theta + t$. We define $\Phi(\theta, t)x = \mathrm{e}^t x$, for all $(t, x, \theta) \in \mathbb{R}_+ \times \mathbb{R}^2$. Then $\pi = (\Phi, \sigma)$ is a linear skew-product flow on $\mathcal{E} = \mathbb{R}^2$ and it is uniformly exponentially expansive.

If $p, q \in [1, \infty)$ with $p < q$ and $\delta \in (p, q)$, we consider the function

$$u : \mathbb{R}_+ \to \mathbb{R}, \quad u(t) = \frac{1}{(t+1)^{1/\delta}}.$$

Then $u \in L^q(\mathbb{R}_+, \mathbb{R}) \setminus L^p(\mathbb{R}_+, \mathbb{R})$. We observe that for $\theta \in \mathbb{R}$, there is no continuous function $f \in L^p(\mathbb{R}_+, \mathbb{R})$, such that the pair (f, u) verifies the equation (E_θ).

Indeed, suppose by contrary that there exists a continuous function $f \in L^p(\mathbb{R}_+, \mathbb{R})$ such that the pair (f, u) satisfies the equation (E_θ). Then

$$\mathrm{e}^{-t} f(t) = \mathrm{e}^{-s} f(s) + \int_s^t \mathrm{e}^{-\tau} u(\tau)\ \mathrm{d}\tau, \quad \forall t \ge s \ge 0.$$

Using the above relation we deduce that

$$f(t) = -\mathrm{e}^t \int_t^\infty \mathrm{e}^{-\tau} u(\tau)\ \mathrm{d}\tau, \quad \forall t \ge 0. \tag{17}$$

But from relation (17) we obtain that

$$\lim_{t\to\infty} \frac{|f(t)|}{u(t)} = \lim_{t\to\infty} \frac{u(t)}{u(t) - u'(t)} = 1.$$

Since $u \notin L^p(\mathbb{R}_+, \mathbb{R})$ it results that $f \notin L^p(\mathbb{R}_+, \mathbb{R})$, which is absurd.

In conclusion, the pair $(L^p(\mathbb{R}_+, \mathbb{R}), L^q(\mathbb{R}_+, \mathbb{R}))$ is not (uniformly) completely admissible for π.

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