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DATA DEPENDENCE FOR THE SOLUTION OF A LOTKA-VOLTERRA SYSTEM WITH TWO DELAYS

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Abstract. The purpose of this paper is to study a Lotka-Volterra system with two delays, by applying fixed point theory.

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1. INTRODUCTION

Let $t, t_0 \in \mathbb{R},\ t < t_0,\ \tau_1, \tau_2 > 0,\ \tau_1 \leq \tau_2,\ f_i \in C([t_0, b] \times \mathbb{R}^4),\ i = 1, 2,$ $\varphi \in C[t_0 - \tau_1, t_0]$, $\psi \in C[t_0 - \tau_2, t_0]$ be given. The problem is to determine

$$x \in C[t_0 - \tau_1, b] \cap C^1[t_0, b]$$
$$y \in C[t_0 - \tau_2, b] \cap C^1[t_0, b]$$

from the Lotka-Volterra systems with two delays

$$\left\{ \begin{array}{l} x'(t) = f_1(t, x(t), y(t), x(t - \tau_1), y(t - \tau_2)) \\ y'(t) = f_2(t, x(t), y(t), x(t - \tau_1), y(t - \tau_2)) \end{array} \right. ,\ t \in [t_0, b],\ t_0 < b \tag{1.1}$$

with initial conditions

$$\left\{ \begin{array}{l} x(t) = \varphi(t),\ t \in [t_0 - \tau_1, t_0] \\ y(t) = \psi(t),\ t \in [t_0 - \tau_2, t_0]. \end{array} \right. \tag{1.2}$$

There have been many studies on this subject (see [2], [5], [8]). The fact that time delays are harmless for the uniform persistence of solutions is established by Wang and Ma for a predator-prey system, by Lu and Takeuchi for competitive systems.

Recently, Saito, Hara and Ma [8] have derived necessary and sufficient conditions for the permanence (uniform persistence) and global stability of a symmetrical Lotka-Volterra-type predator-prey system with two delays.

For a nonautonomous competitive Lotka-Volterra system with no delays, recently Ahmad and Lazer have established the average conditions for the persistence, which are weaker than those of Gopalsamy, Tineo and Alvarez for periodic or almost-periodic cases.

Here we study the existence and uniqueness of the solution using the contraction principle and the data dependence using Lemma 2 for the problem (1.1)+(1.2).

2. EXISTENCE AND UNIQUENESS

The purpose of this section is to find the conditions for the existence and uniqueness of the solution of problem (1.1)+(1.2). Let (x, y) be a solution of (1.1)+(1.2). The problem (1.1)+(1.2) is equivalent to

(2.1)
$$x(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau_1, t_0] \\ \varphi(t_0) + \int_{t_0}^{t} f_1(s, x(s), y(s), x(s - \tau_1), y(s - \tau_2))\mathrm{d}s, & t \in [t_0, b] \end{cases}$$

(2.2)
$$y(t) = \begin{cases} \psi(t), & t \in [t_0 - \tau_2, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f_2(s, x(s), y(s), x(s - \tau_1), y(s - \tau_2))\mathrm{d}s, & t \in [t_0, b] \end{cases}$$

where $x \in C[t_0 - \tau_1, b]$ and $y \in C[t_0 - \tau_2, b]$.

We consider the operator $A_f : C[t_0 - \tau_1, b] \times C[t_0 - \tau_2, b] \to C[t_0 - \tau_1, b] \times C[t_0 - \tau_2, b]$ and we remark that it follows

$$(x, y) = A_f(x, y), \tag{2.3}$$

where

$$\begin{aligned} A_f(x, y)(t) = \Bigg(& \varphi(t_0) + \int_{t_0}^{t} f_1(s, x(s), y(s), x(s - \tau_1), y(s - \tau_2))\mathrm{d}s, \\ & \psi(t_0) + \int_{t_0}^{t} f_2(s, x(s), y(s), x(s - \tau_1), y(s - \tau_2))\mathrm{d}s \Bigg). \end{aligned} \tag{2.4}$$

Consider the Banach space $C[t_0, b]$ with Bielecki norm $\|\cdot\|_B$ defined by

$$\|x\|_B = \max_{t_0 \le t \le b} |x(t)| \, \mathrm{e}^{-\rho(t - t_0)}, \ \rho > 0. \tag{2.5}$$

For $t \in [t_0 - \tau_1, t_0]$ we have $|A_f(x, y)(t) - A_f(\overline{x}, \overline{y})(t)| = 0$.

For $t \in [t_0 - \tau_2, t_0]$ we have $|A_f(x, y)(t) - A_f(\overline{x}, \overline{y})(t)| = 0$.

For $t \in [t_0, b]$, let (X, d) be a metric space with $X = (C[t_0, b], \|\cdot\|_B)$ and $(x, y), (\overline{x}, \overline{y}) \in X \times X$, then:

$$\begin{aligned} d(A_f(x, y), A_f(\overline{x}, \overline{y})) &= |\pi_1 A_f(x, y)(t) - \pi_1 A_f(\overline{x}, \overline{y})(t)| \\ &= \Bigg| \varphi(t_0) + \int_{t_0}^{t} f_1(s, x(s), y(s), x(s - \tau_1), y(s - \tau_2))\mathrm{d}s - \varphi(t_0) \\ &\quad - \int_{t_0}^{t} f_1(s, \overline{x}(s), \overline{y}(s), \overline{x}(s - \tau_1), \overline{y}(s - \tau_2))\mathrm{d}s \Bigg| \\ &\le L \Bigg[\int_{t_0}^{t} |x(s) - \overline{x}(s)| \, \mathrm{e}^{-\rho(s - t_0)} \mathrm{e}^{\rho(s - t_0)} \mathrm{d}s \\ &\quad + \int_{t_0}^{t} |y(s) - \overline{y}(s)| \, \mathrm{e}^{-\rho(s - t_0)} \mathrm{e}^{\rho(s - t_0)} \mathrm{d}s \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
&+\int_{t_0}^{t} |x(s-\tau_1)-\overline{x}(s-\tau_1)|\, \mathrm{e}^{-\rho(s-\tau_1-t_0)}\mathrm{e}^{\rho(s-\tau_1-t_0)}\mathrm{d}s \\
&+\int_{t_0}^{t} |y(s-\tau_2)-\overline{y}(s-\tau_2)|\, \mathrm{e}^{-\rho(s-\tau_2-t_0)}\mathrm{e}^{\rho(s-\tau_2-t_0)}\mathrm{d}s\Big] \\
\le{}& L\left(2\left\|x-\overline{x}\right\|_B \frac{1}{\rho}\mathrm{e}^{\rho(t-t_0)} + 2\left\|y-\overline{y}\right\|_B \frac{1}{\rho}\mathrm{e}^{\rho(t-t_0)}\right) \\
\le{}& \frac{2L}{\rho}\mathrm{e}^{\rho(t-t_0)}\left(\left\|x-\overline{x}\right\|_B + \left\|y-\overline{y}\right\|_B\right).
\end{aligned}
$$

Consequently,

$$\left\|\pi_1 A_f(x,y)-\pi_1 A_f(\overline{x},\overline{y})\right\|_B \le \frac{2L}{\rho} d((x,y),(\overline{x},\overline{y})),$$

where π_1 is the first projection for $A_f(x,y)$ from (2.4).

By similar calculations we obtain

$$\left\|\pi_2 A_f(x,y)-\pi_2 A_f(\overline{x},\overline{y})\right\|_B \le \frac{2L}{\rho} d((x,y),(\overline{x},\overline{y})), \tag{2.7}$$

where π_2 is the second projection for $A_f(x,y)$ from (2.4). We deduce

$$
\begin{aligned}
d(A_f(x,y),A_f(\overline{x},\overline{y})) &= \left\|\pi_1 A(x,y)-\pi_1 A(\overline{x},\overline{y})\right\|_B \\
&\quad + \left\|\pi_2 A(x,y)-\pi_2 A(\overline{x},\overline{y})\right\|_B \\
&\le \frac{4L}{\rho} d((x,y),(\overline{x},\overline{y})).
\end{aligned} \tag{2.8}
$$

Then A_f is Lipschitz with a Lipschitz constant $L_{A_f}=\frac{4L}{\rho}$. For $\rho=4L+1$, A_f is a contraction. By the contraction principle we have:

Theorem 1. *We suppose that:*
(i) $f_i\in C([t_0,b]\times\mathbb{R}^4),\ i=1,2$;
(ii) *there is* $L>0$ *such that*

$$
\begin{aligned}
&|f_i(t,u_1,u_2,u_3,u_4)-f_i(t,v_1,v_2,v_3,v_4)| \\
&\qquad\le L(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|+|u_4-v_4|),
\end{aligned}
$$

for all $t\in[t_0,b],\ u_i,v_i\in\mathbb{R},\ i=\overline{1,4}$;
(iii) $\frac{4L}{4L+1}<1$.

Then the problem (1.1)+(1.2) *has in* $C[t_0-\tau_1,b]\times C[t_0-\tau_2,b]$ *a unique solution. Moreover, if* (x^*,y^*) *the unique solution of* (1.1)+(1.2), *then*

$$(x^*,y^*)=\lim_{n\to\infty} A_f^n(x,y),\ \textit{for all } x\in C[t_0-\tau_1,b],\ y\in C[t_0-\tau_2,b].$$

3. DATA DEPENDENCE

In this section we shall discus a theorem of data dependence for the solution of problem (1.1)+(1.2). To prove data dependence relation we need the following lemma:

LEMMA 2 (I.A. Rus). *Let* (X,d) *be a complete metric space and* $A, B : X \to X$ *two operators. We suppose that:*

(i) A *is an* α*-contraction;*

(ii) *there is* $\eta > 0$ *such that* $d(A(x), B(x)) \leq \eta$, $\forall x \in X$*;*

(iii) $x_B^* \in F_B$.

Then $d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}$, *where* x_A^* *is the unique fixed point of* A.

We have:

THEOREM 3. *Let* $f_1^1,\ f_2^1,\ f_1^2,\ f_2^2,\ \varphi^1,\ \varphi^2,\ \psi^1,\ \psi^2$ *be under the hypothesis of* Theorem 1. *We suppose that there exist* $\eta_i > 0$, $i = 1,2,3$, *such that*

$$\left|\varphi^1(t) - \varphi^2(t)\right| \leq \eta_1,\ \forall t \in [t_0 - \tau_1, t_0],$$

$$\left|\psi^1(t) - \psi^2(t)\right| \leq \eta_2,\ \forall t \in [t_0 - \tau_2, t_0]$$

and

$$\left|f_i^1(t,u_1,u_2,u_3,u_4) - f_i^2(t,u_1,u_2,u_3,u_4)\right| \leq \eta_3,$$

for all $t \in [t_0, b]$, $u_1, u_2, u_3, u_4 \in \mathbb{R}$. *Then*

$$\|(x_1^*, y_1^*) - (x_2^*, y_2^*)\|_B \leq \frac{\eta_1 + \eta_2 + 2\eta_3(t - t_0)}{1 - \frac{4L}{4L+1}},$$

where (x_i^*, y_i^*), $i = 1,2$, *are solutions of the problems* (1.1)+(1.2) *with data* f_i^1, φ^1, ψ^1, *respectively with data* f_i^2, φ^2, ψ^2, $i = 1,2$.

Proof. Consider that we are under the hypothesis of Theorem 1. If (x_1^*, y_1^*) is a solution of problem (1.1)+(1.2) with data $f_i^1, \varphi^1, \psi^1, A_f^1$, $i = 1,2$, and if (x_2^*, y_2^*) is a solution of problem (1.1)+(1.2) with data $f_i^2, \varphi^2, \psi^2, A_f^2$, $i = 1,2$, then it follows that

$$\begin{aligned}
&\left|\pi_1 A_f^1(x,y)(t) - \pi_1 A_f^2(x,y)(t)\right| \\
&= \Big|\varphi^1(t_0) + \int_{t_0}^t f_1^1(s, x(s), y(s), x(s-\tau_1), y(s-\tau_2))\mathrm{d}s \\
&\quad -\varphi^2(t_0) - \int_{t_0}^t f_1^2(s, x(s), y(s), x(s-\tau_1), y(s-\tau_2))\mathrm{d}s\Big| \\
&\leq \left|\varphi^1(t_0) - \varphi^2(t_0)\right| + \int_{t_0}^t \big|f_1^1(s, x(s), y(s), x(s-\tau_1), y(s-\tau_2)) \\
&\quad - f_1^2(s, x(s), y(s), x(s-\tau_1), y(s-\tau_2))\big|\,\mathrm{d}s \leq \eta_1 + \eta_3(t - t_0).
\end{aligned} \tag{3.1}$$

We have

$$\left|\pi_1 A_f^1(x,y)(t) - \pi_1 A_f^2(x,y)(t)\right| \leq \eta_1 + \eta_3(t - t_0), \tag{3.2}$$

$$\left|\pi_1 A_f^1(x,y)(t) - \pi_1 A_f^2(x,y)(t)\right| \mathrm{e}^{-\rho(t-t_0)} \leq \eta_1 + \eta_3(t - t_0), \tag{3.3}$$

$$\left\|\pi_1 A_f^1(x,y)(t) - \pi_1 A_f^2(x,y)(t)\right\|_B \leq \eta_1 + \eta_3(t - t_0). \tag{3.4}$$

Analogously

$$\left\|\pi_2 A_f^1(x,y)(t) - \pi_2 A_f^2(x,y)(t)\right\| \le \eta_2 + \eta_3(t - t_0).$$

Then

$$\left\|A_f^1(x,y) - A_f^2(x,y)\right\|_B \le \eta_1 + \eta_2 + 2\eta_3(t - t_0).$$

From Lemma 2 we have:

$$\|(x_1^*, y_1^*) - (x_2^*, y_2^*)\|_B \le \frac{\eta_1 + \eta_2 + 2\eta_3(t - t_0)}{1 - \frac{4L}{4L+1}}.$$

So the proof is complete. □

4. EXAMPLES

Let $\mu \in \mathbb{R},\ \varphi \in C[-1,0],\ \psi \in C[-2,0]$ be given. We consider the problem

$$\text{(4.1)} \qquad \begin{cases} x'(t) = \mu[x(t-1) + y(t-2)], & t \in [0,2] \\ y'(t) = \mu[-x(t-1) - y(t-2)], & t \in [0,2] \\ x(t) = \varphi(t), & t \in [-1,0] \\ y(t) = \psi(t), & t \in [-2,0]. \end{cases}$$

Then

$$\text{(4.2)} \qquad x(t) = \begin{cases} \varphi(t), & t \in [-1,0] \\ \varphi(t_0) + \displaystyle\int_0^t \mu[x(s-1) + y(s-2)]\mathrm{d}s, & t \in [0,2] \end{cases}$$

$$\text{(4.3)} \qquad y(t) = \begin{cases} \psi(t), & t \in [-2,0] \\ \psi(t_0) + \displaystyle\int_0^t \mu[-x(s-1) - y(s-2)]\mathrm{d}s, & t \in [0,2] \end{cases}$$

Note that if we take $A : C[-1,2] \times [-2,2] \to C[-1,2] \times [-2,2]$ defined by

$$A(x,y)(t) = \Bigg(\varphi(t_0) + \int_0^t \mu[x(s-1) + y(s-2)]\mathrm{d}s\ ,$$
$$\psi(t_0) + \int_0^t \mu[-x(s-1) - y(s-2)]\mathrm{d}s\Bigg),$$

then the problem (4.1) is equivalent to

$$(x,y) = A(x,y).$$

From Theorem 1 the problem (4.1) has a unique solution.

In what follows we discuss the data dependence of the solution.

Let $\varphi^1,\ \varphi^2,\ \psi^1,\ \psi^2$. We suppose that there are $\delta_i > 0$, $i = 1,2,3$, such that

$$\left|\varphi^1(t) - \ \varphi^2(t)\right| < \delta_1,$$
$$\left|\psi^1(t) - \ \psi^2(t)\right| < \delta_2,$$
$$\left|\mu^1 - \mu^2\right| |x(t-1) + y(t-2)\ | < \delta_3.$$

Let us consider the problems:

$$(4.4)\qquad \begin{cases} x'(t) = \mu^1[x(t-1) + y(t-2)], & t \in [0,2] \\ y'(t) = \mu^1[-x(t-1) - y(t-2)] & \\ x(t) = \varphi^1(t), & t \in [-1,0] \\ y(t) = \psi^1(t), & t \in [-2,0] \end{cases}$$

$$(4.5)\qquad \begin{cases} x'(t) = \mu^2[x(t-1) + y(t-2)] & t \in [0,2] \\ y'(t) = \mu^2[-x(t-1) - y(t-2)] & \\ x(t) = \varphi^2(t), & t \in [-1,0] \\ y(t) = \psi^2(t), & t \in [-2,0] \end{cases}$$

If (x_1^*, y_1^*) is a solution for the problem (4.4) and (x_2^*, y_2^*) is a solution for the problem (4.5), we look for a estimation of $\|(x_1^*, y_1^*) - (x_2^*, y_2^*)\|$. We have the operators $A_f^1(x,y)(t)$ and $A_f^2(x,y)(t)$. It follows that

$$\left\|A_f^1(x,y) - A_f^2(x,y)\right\| \le \delta_1 + \delta_2 + 2\delta_3.$$

From Theorem 3, we have

$$\|(x_1^*, y_1^*) - (x_2^*, y_2^*)\| \le \frac{\delta_1 + \delta_2 + 2\delta_3}{1 - \frac{4L}{4L+1}}.$$

5. REMARKS AND GENERALIZATIONS

Remark 1. *Theorems* 1 *and* 3 *also hold if we make some changes on the arguments as follows: instead of* $x(t-\tau_1)$ *we put* $g_1(t)$ *with* $g_1 \in C([t_0,b],[t_0-\tau_1,t_0])$, *and instead of* $y(t-\tau_2)$ *we have* $g_2(t)$ *with* $g_2 \in C([t_0,b],[t_0-\tau_2,t_0])$.

Remark 2. *Let* $f \in C([t_0,b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $\varphi_i \in C([t_0-\tau_i,t_0],\mathbb{R}^n)$, $i = 1,2,,\ldots,n$, $t_0, t \in \mathbb{R}$, $t_0 < t$, $\tau_1, \tau_2, \ldots, \tau_n > 0$, $\tau_1 < \tau_2 < \cdots < \tau_n$. *We extend the same discussion to n populations, with the specification that the populations are in the same environment – prade or predator.*

Let $x_1(t), x_2(t), \ldots, x_n(t)$ be lows of growing, continuous and derivable. Then we have the system

$$(5.1)\qquad \begin{cases} x_1'(t) = f_1(t, x_1(t), \ldots, x_n(t), x_1(t-\tau_1), \ldots, x_n(t-\tau_n)) \\ x_2'(t) = f_2(t, x_1(t), \ldots, x_n(t), x_1(t-\tau_1), \ldots, x_n(t-\tau_n)) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n'(t) = f_n(t, x_1(t), \ldots, x_n(t), x_1(t-\tau_1), \ldots, x_n(t-\tau_n)), \end{cases}$$

where $t \in [t_0,b]$, $i = 1,\ldots,n$, $f_i \in C([t_0,b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, $i = 1,\ldots,n$, and the initial conditions

$$(5.2)\qquad \begin{cases} x_1(t) = \varphi_1(t),\ t \in [t_0-\tau_1, t_0] \\ x_2(t) = \varphi_2(t),\ t \in [t_0-\tau_2, t_0] \\ \cdots\cdots\cdots\cdots\cdots \\ x_n(t) = \varphi_n(t),\ t \in [t_0-\tau_n, t_0] \end{cases}$$

The problem is to determine $x_i \in C([t_0-\tau_i,b])\cap C^1[t_0,b],\ i=1,\dots,n$, that suits the problem (5.1)+(5.2).

By the contraction principle we have:

THEOREM 4. *Assume that the following conditions hold:*
(i) *there is $L>0$ such that*

$$\begin{aligned}&|f_i(t,u_{11},\dots,u_{1n},v_{11},\dots,v_{1n})-f_i(t,u_{21},\dots,u_{2n},v_{21},\dots,v_{2n})|\\ &\le L(|u_{11}-u_{21}|+\dots+|u_{1n}-u_{2n}|+|v_{11}-v_{21}|+\dots+|v_{1n}-v_{2n}|),\end{aligned}$$

for all $t\in[t_0,b],\ u_{ji},v_{ji}\in\mathbb{R},\ i=1,2,\dots,n,\ j=1,2;$
(ii) $\frac{2nL}{2nL+1}<1$.
Then the problem (5.1)+(5.2) *has a unique solution. Moreover, if the unique solution of* (5.1)+(5.2) *is $(x_1^*,\dots,x_n^*)$, then*

$$(x_1^*,\dots,x_n^*)=\lim_{n\to\infty}A_f^n(x_1,\dots,x_n),\ \textit{for all } x_i\in C[t_0-\tau_i,b],\ i=1,2,\dots,n.$$

Applying Lemma 2 we have:

THEOREM 5. *Let $f_i^k,\ \varphi_i^k, k=1,2,\ i=1,\dots,n$, satisfying the hypotheses of Theorem 4. We assume that there exist $\eta_i^k>0,\ k=1,2,\ i=1,\dots,n$, such that*

$$\left|\varphi_i^1(t)-\varphi_i^2(t)\right|\le\eta_i^1,\ \forall t\in[t_0-\tau_i,t_0],\quad i=1,2,\dots,n$$

and

$$\left|f_i^1(t,u_1,\dots,u_n,v_1,\dots,v_n)-f_i^2(t,u_1,\dots,u_n,v_1,\dots,v_n)\right|\le\eta_i^2$$

for all $t\in[t_0,b],\ u_i,v_i\in\mathbb{R}, i=1,2,\dots,n$. Then

$$\left\|(x_1^{1*},\dots,x_n^{1*})-(x_1^{2*},\dots,x_n^{2*})\right\|_B\le\frac{\eta_1^1+\dots+\eta_n^1+(\eta_1^2+\dots+\eta_n^2)(t-t_0)}{1-\frac{2nL}{2nL+1}},$$

where $(x_1^{k*},\dots,x_n^{k*}),\ k=1,2$, are solutions of the problems (5.1)+(5.2) *with data f_i^1,φ_i^1, and f_i^2,φ_i^2 respectively.*

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