MATHEMATICA, Tome 48 (71), N° 1, 2006, pp. 13–18

GEOMETRIC PROPERTIES OF GENERALIZED BESSEL FUNCTIONS OF COMPLEX ORDER

ÁRPÁD BARICZ

Abstract. In this paper we obtain conditions of univalence and convexity for the generalized and normalized Bessel functions of the first kind of complex order using the technique of differential subordinations. A condition of starlikeness of $zu_p(z)$ is given, where by definition

$$u_p(z) := \sum_{n=0}^{\infty} \left(-\frac{c}{4}\right)^n \frac{\Gamma\left(p+\frac{b+1}{2}\right)}{\Gamma\left(p+n+\frac{b+1}{2}\right)} \frac{z^n}{n!},\ \ b,p,c,z \in \mathbb{C}.$$

MSC 2000. 30C45, 30A20.

Key words. Differential subordination, convex functions, univalent functions, starlike functions, Bessel functions.

1. INTRODUCTION AND PRELIMINARIES

A function f, analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, is said to be *convex* if it is univalent and $f(U)$ is a convex domain. It is well known that f is convex [4] if and only if $f'(0) \neq 0$ and

$$\operatorname{Re}\left[1 + zf''(z)/f'(z)\right] > 0,\ \ z \in U.$$

A function g, analytic in U, with $g(0) = 0$, is said to be *starlike* if it is univalent and $g(U)$ is starlike with respect to the origin. The function g with $g(0) = 0$ and $g'(0) \neq 0$ is starlike [4] if and only if

$$\operatorname{Re}\left[zg'(z)/g(z)\right] > 0,\ \ z \in U.$$

If in addition

$$\operatorname{Re}\left[zg'(z)/g(z)\right] > \alpha,\ \ z \in U,$$

where $0 \leq \alpha < 1$, then g is called *starlike of order* α. We remark that, according to the Alexander duality Theorem [1], the function f is convex if and only if zf' is starlike.

The next lemmas will be used to prove several theorems.

LEMMA 1.1. [6] *Let Ω be a set in the complex plane $\mathbb{C}$ and $\psi :\ \mathbb{C}^3 \times U \mapsto \mathbb{C}$ a function, that satisfies the admissibility condition $\psi(\rho\mathrm{i}, \sigma, \mu + \nu\mathrm{i}; z) \notin \Omega$, where $z \in U$, $\rho, \sigma, \mu, \nu \in \mathbb{R}$ with $\mu + \sigma \leq 0$ and $\sigma \leq -(1 + \rho^2)/2$. If H is analytic in*

This work was supported partially by the Institute of Mathematics, University of Debrecen, Hungary. The author thanks Professors Péter T. Nagy and Saminathan Ponnusamy for their support and encouragement, and is grateful to the referee for his/her remarks, which were very valuable.

the unit disk U, *with* $H(0)=1$ *and* $\psi\left(H(z), zH'(z), z^2H''(z); z\right) \in \Omega,\ z \in U$, *then* $\mathrm{Re}[H(z)] > 0, \forall\ z \in U$.

This lemma is a special case of [6, Theorem 2.3b], obtained by taking $q(z) = (1+z)/(1-z)$. If we only have $\psi : \mathbb{C}^2 \times U \mapsto \mathbb{C}$, the *admissibility condition* reduces to $\psi(\rho\mathrm{i}, \sigma; z) \notin \Omega,\ z \in U$ and $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1+\rho^2)/2$. We continue these preliminaries with the next conditions of univalence (close-to-convexity [4]) due to Ozaki [7].

Lemma 1.2. [7] *Let* D *be a simply connected domain and let* f *an analytic function in* D. *If there exists a function* φ, *univalent in* D *such that* $\varphi(D)$ *is a convex domain and* $\mathrm{Re}[f'(z)/\varphi'(z)] > 0$, *for all* $z \in D$, *i.e.* f *is close-to-convex, then* f *is univalent in* D.

We recall that the generalized Bessel function v_p of real order [3] is defined as a particular solution of the linear differential equation

$$z^2v''(z) + bzv'(z) + \left[cz^2 - p^2 + (1-b)p\right]v(z) = 0, \tag{1.1}$$

where $b, p, c \in \mathbb{R}$. The analytic function v_p has the form

$$v_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!\Gamma\left(p+n+\frac{b+1}{2}\right)} \cdot \left(\frac{z}{2}\right)^{2n+p}, \quad z \in \mathbb{C}. \tag{1.2}$$

Let us consider now $b, p, c \in \mathbb{C}$ in (1.1).

Definition 1.3. Any solution of the linear differential equation (1.1) is called a generalized Bessel function of complex order p and the particular solution v_p defined by (1.2) is called the *generalized Bessel function of the first kind of complex order* p.

Now, the generalized and normalized Bessel function u_p is defined with the transformation $u_p(z) = [a_0(p)]^{-1}z^{-p/2}v_p(z^{1/2})$, where

$$a_0 = \left[2^p\Gamma\left(p + \frac{b+1}{2}\right)\right]^{-1} \equiv a_0(p).$$

Using the Pochhammer symbol, defined, in terms of Γ-functions, by $(\kappa)_n = \Gamma(\kappa+n)/\Gamma(\kappa) = \kappa(\kappa+1)\ldots(\kappa+n-1)$ and $(\kappa)_0 = 1$, we obtain for the function u_p the following form

$$u_p(z) = \sum_{n\geq 0} \frac{(-1)^n c^n}{4^n (\kappa)_n} \frac{z^n}{n!} = \sum_{n\geq 0} \left(-\frac{c}{4}\right)^n [(\kappa)_n]^{-1} \frac{z^n}{n!}, \tag{1.3}$$

where $\kappa = p + (b+1)/2 \neq 0, -1, -2, \ldots$. The function u_p is called the generalized and normalized Bessel function of the first kind of complex order p, this function is analytic in $\mathbb{C}$ and satisfies the differential equation

$$4z^2u''(z) + 2(2p+1+b)zu'(z) + czu(z) = 0. \tag{1.4}$$

Remark 1.4. By Proposition 2.17, [3] we know that $2(2p+b+1)u_p'(z) = -cu_{p+1}(z)$, or $4\kappa u_p'(z) = -cu_{p+1}(z)$, with $\kappa = p+(b+1)/2$ and for all $b,p,c \in \mathbb{R}$. Clearly this recursive relation remains true for $b,p,c \in \mathbb{C}$.

2. UNIVALENCE, CONVEXITY AND STAR-LIKENESS OF BESSEL FUNCTIONS

Theorem 2.1. *For b,p, $c = c_1 + \mathrm{i}c_2 \in \mathbb{C}$ and $\kappa = p+(b+1)/2$ the Bessel functions, v_p and u_p, satisfy the following properties in the unit disk U:*

(i) *If $\operatorname{Re}\kappa \geq |c|/4+1$, then $\operatorname{Re} u_p(z) > 0$;*
(ii) *For $\operatorname{Re}\kappa \geq |c|/4$ we have that u_p is univalent;*
(iii) *For $\operatorname{Re}\kappa \geq |c|/4 + (2\operatorname{Im}\kappa - 1)^2/24 + 1/2$ we have that u_p is convex;*
(iv) *If $\operatorname{Re}\kappa \geq |c|/4 + (2\operatorname{Im}\kappa - 1)^2/24 + 3/2$, then $zu_p(z)$ is starlike;*
(v) *If $\operatorname{Re}\kappa \geq |c|/2 + (2\operatorname{Im}\kappa - 1)^2/16 + 1$, then $zu_p(z)$ is starlike of order $1/2$;*
(vi) *For $\operatorname{Re}\kappa \geq |c|/2 + (2\operatorname{Im}\kappa - 1)^2/16 + 1$ we have that $z^{1-p}v_p(z)$ is starlike.*

Proof. (i) Denoting $H(z) = u_p(z)$, since H satisfies (1.4), it will also satisfy the following differential equation

$$4z^2H''(z) + 4\kappa zH'(z) + czH(z) = 0. \tag{2.1}$$

Letting $\psi(r,s,t;z) = 4t+4\kappa s+czr$ and $\Omega = \{0\}$, equation (2.1) can be written as $\psi\left(H(z), zH'(z), z^2H''(z); z\right) \in \Omega$. Now we will use Lemma 1.1 to prove that $\operatorname{Re}[H(z)] > 0$. If we let $z = x + \mathrm{i}y$, then

$$\operatorname{Re}\psi\left(\rho\mathrm{i}, \sigma, \mu+\nu\mathrm{i}; x+\mathrm{i}y\right) = 4(\mu+\sigma) + 4(\operatorname{Re}\kappa - 1)\sigma - (c_1y + c_2x)\rho.$$

For $\operatorname{Re}\kappa > 1$ we have that

$$\operatorname{Re}\psi\left(\rho\mathrm{i}, \sigma, \mu+\nu\mathrm{i}; x+\mathrm{i}y\right) \leq -2(\operatorname{Re}\kappa - 1)\rho^2 - (c_1y + c_2x)\rho - 2(\operatorname{Re}\kappa - 1),$$

and denoting

$$Q_1(\rho) = -2(\operatorname{Re}\kappa - 1)\rho^2 - (c_1y + c_2x)\rho - 2(\operatorname{Re}\kappa - 1),$$

this will be negative for all real ρ, because the discriminant Δ_1 of $Q_1(\rho)$ satisfies

$$\Delta_1 = (c_1y + c_2x)^2 - 16(\operatorname{Re}\kappa - 1)^2 < 0,$$

whenever $x, y \in (-1,1)$ and $\operatorname{Re}\kappa \geq |c|/4+1$. Suppose that $\Delta_1 \geq 0$. This is equivalent to $4(\operatorname{Re}\kappa - 1) \leq |c_1y + c_2x|$, but using the Cauchy-Schwarz-Buniakowski inequality and the hypothesis we have that

$$4(\operatorname{Re}\kappa - 1) \leq |c_1y + c_2x| \leq \sqrt{c_1^2 + c_2^2}\sqrt{x^2+y^2} < |c| \leq 4(\operatorname{Re}\kappa - 1),$$

and this is contradiction, therefore $\Delta_1 < 0$. Hence by Lemma 1.1 we conclude that $\operatorname{Re}[H(z)] = \operatorname{Re}[u_p(z)] > 0$, for all $z \in U$.

(ii) If we apply (i) and Lemma 1.2 for the special case $D = U$, $\varphi(z) = -(cz)/(4\kappa)$, we obtain the univalence condition. Therefore, if $\operatorname{Re}\kappa \geq |c|/4$, we obtain that $\operatorname{Re} u_{p+1}(z) > 0$, for all $z \in U$. Using Remark 1.4 we conclude that

$$\operatorname{Re} u_{p+1}(z) = \operatorname{Re}\left[-\frac{4\kappa}{c}u_p'(z)\right] > 0, \forall\ z \in U, \tag{2.2}$$

which means that u_p is close-to-convex of order 0, i.e. it is univalent in U.

(iii) Denoting by

$$H(z) = 1 + \frac{zu_p''(z)}{u_p'(z)}, \tag{2.3}$$

for $2(2\operatorname{Re}\kappa - 1) \geq |c| + (2\operatorname{Im}\kappa - 1)^2/6 > |c| - 2$, the condition of (ii) holds, hence $u_p'(z) \neq 0$ and H is analytic in U with $H(0) = 1$. Combining (2.3) with (1.4), we obtain that H satisfies the next differential equation

$$4zH'(z) + 4H^2(z) + 4(\kappa - 2)H(z) + cz - 4(\kappa - 1) = 0. \tag{2.4}$$

If we let $\psi(r, s; z) = 4s + 4r^2 + 4(\kappa - 2)r + cz - 4(\kappa - 1)$ and $\Omega = \{0\}$, then (2.4) can be written as $\psi(H(z), zH'(z); z) \in \Omega$. Now we will use Lemma 1.1 to prove that $\operatorname{Re} H(z) > 0$. Letting $z = x + \mathrm{i}y$ and $c = c_1 + \mathrm{i}c_2$, we obtain

$$\begin{aligned}\operatorname{Re}\psi(\rho\mathrm{i}, \sigma; x + \mathrm{i}y) &= 4\sigma - 4\rho^2 - 2\rho(2\operatorname{Im}\kappa - 1) + (c_1x - c_2y) - 4(\operatorname{Re}\kappa - 1)\\ &\leq -6\rho^2 - 2(2\operatorname{Im}\kappa - 1)\rho + (c_1x - c_2y) - 2(2\operatorname{Re}\kappa - 1) = Q_2(\rho),\end{aligned}$$

for $\sigma \leq -(1 + \rho^2)/2$, for all real ρ and for $x, y \in (-1, 1)$. The discriminant Δ_2 of the quadratic form $Q_2(\rho)$, is the following:

$$\Delta_2 = 4[(2\operatorname{Im}\kappa - 1)^2 + 6(c_1x - c_2y) - 12(2\operatorname{Re}\kappa - 1)]. \tag{2.5}$$

It is easy to check that $c_1x - c_2y \leq |c_1| + |c_2|$, for any $x, y \in (-1, 1)$ and for $c_1, c_2 \in \mathbb{R}$. Otherwise by the Cauchy-Schwarz-Buniakowski inequality we have that $c_1x - c_2y \leq |c_1x - c_2y| \leq \sqrt{c_1^2 + c_2^2}\sqrt{x^2 + y^2} < |c|$ and clearly $|c| \leq |c_1| + |c_2|$. Therefore we have that

$$\Delta_2/4 < (2\operatorname{Im}\kappa - 1)^2 + 6|c| - 12(2\operatorname{Re}\kappa - 1), \tag{2.6}$$

which, by hypothesis, is negative. Thus, the quadratic form $Q_2(\rho)$ is also negative, which means that $\operatorname{Re}\psi(\rho\mathrm{i}, \sigma; x + \mathrm{i}y) < 0$. Then we conclude that

$$\operatorname{Re} H(z) = \operatorname{Re}\left[1 + \frac{zu_p''(z)}{u_p'(z)}\right] > 0, \forall\ z \in U, \tag{2.7}$$

which shows that u_p is convex in U.

(iv) We have $czu_p(z) = -4\kappa zu_{p-1}'(z)$ by Remark 1.4 for $p-1$ and the result follows immediately by applying (iii) and Alexander's duality Theorem [1]. Since $\operatorname{Re}\kappa - 1 \geq |c|/4 + 1/2 + (2\operatorname{Im}\kappa - 1)^2/24$ and using the fact that u_{p-1} is convex, it follows that zu_{p-1}' is starlike in the unit disk.

(v) Let denote $G_p(z) = zu_p(z)$. Since the condition of the first part holds, i.e.

$$\operatorname{Re}\kappa \geq |c|/2 + (2\operatorname{Im}\kappa - 1)^2/16 + 1 > |c|/4 + 1,$$

we deduce that $G_p(z) \neq 0$, $\forall z \in U$. If we set

$$H(z) = 2\frac{zG_p'(z)}{G_p(z)} - 1 = 1 + 2\frac{zu_p'(z)}{u_p(z)}, \tag{2.8}$$

then H is analytic in U, with $H(0) = 1$. Combining (2.8) with (1.4) we obtain the equation

$$2zH'(z) + H^2(z) + 2(\kappa - 2)H(z) + cz - (2\kappa - 3) = 0. \tag{2.9}$$

If we let

$$\psi(r, s; z) = 2s + r^2 + 2(\kappa - 2)r + cz - (2\kappa - 3)$$

and $\Omega = \{0\}$, then (2.9) can be written as $\psi\left(H(z), zH'(z); z\right) \in \Omega$. We will use Lemma 1.1 to prove that $\operatorname{Re} H(z) > 0$, $z \in U$. Letting $z = x + \mathrm{i}y$ and $c = c_1 + \mathrm{i}c_2$, we obtain

$$\begin{aligned}\operatorname{Re}\psi(\rho\mathrm{i}, \sigma; x + \mathrm{i}y) &= 2\sigma - \rho^2 - (2\operatorname{Im}\kappa - 1)\rho + (c_1x - c_2y) - (2\operatorname{Re}\kappa - 3)\\ &\leq -2\rho^2 - (2\operatorname{Im}\kappa - 1)\rho + (c_1x - c_2y) - (2\operatorname{Re}\kappa - 2) = Q_3(\rho),\end{aligned}$$

for $\sigma \leq -(1 + \rho^2)/2$, for all real ρ and for $x, y \in (-1, 1)$. An analogous procedure gives the proof of $Q_3(\rho) < 0$ under the assumptions. We obtain that the discriminant Δ_3 of the quadratic form $Q_3(\rho)$ is

$$\Delta_3 = (2\operatorname{Im}\kappa - 1)^2 + 8(c_1x - c_2y) - 8(2\operatorname{Re}\kappa - 2). \tag{2.10}$$

We know, by Cauchy's inequality, that $c_1x - c_2y < |c|$, therefore by hypothesis we have that $\Delta_3 < (2\operatorname{Im}\kappa - 1)^2 + 8|c| - 8(2\operatorname{Re}\kappa - 2) \leq 0$. Hence we conclude that

$$\operatorname{Re} H(z) = \operatorname{Re}\left[2\frac{zG_p'(z)}{G_p(z)} - 1\right] > 0, \forall\ z \in U, \tag{2.11}$$

which shows that $G_p(z) = zu_p(z)$ is starlike of order $1/2$.

(vi) If we let $H_p(z) = z^{1-p}v_p(z)$, then $H_p(z) = G_p(z^2)/z = zu_p(z^2)$. Since $G_p(z) = zu_p(z)$ is starlike of order $1/2$ and

$$\operatorname{Re}\left[\frac{zH_p'(z)}{H_p(z)}\right] = \operatorname{Re}\left[2\frac{z^2G_p'(z^2)}{G_p(z^2)} - 1\right] > 0, \forall\ z \in U, \tag{2.12}$$

we deduce that H_p is starlike in U. $\square$

Remark 2.2. Note that similar results as in Theorem 2.1 for confluent hypergeometric functions was obtained by S. Kanas and J. Stankiewicz [5]. In the case of real b, p, c, we obtain that $2\operatorname{Im}\kappa - 1 = 0$, therefore Theorem 2.1 reduces to the results in [2] (see also [3, Theorem 3.1]).

REFERENCES

[1] Alexander, J. W., *Functions wich map the interior of the unit circle upon simple regions*, Ann. Math., **17** (1915), 12–29.

[2] Baricz, Á., *Applications of the admissible functions method for some differential equations*, Pure Math. Appl., **13** (4) (2002), 433–440.

[3] Baricz, Á., *Geometric properties of generalized Bessel functions*, J. Math. Anal. Appl., submitted.

[4] Duren, P. L., *Univalent Functions*, Grundlehren Math. Wiss., **259** (1983), Springer-Verlag, New York.

[5] Kanas, S. and Stankiewicz, J., *Univalence of confluent hypergeometric functions*, Ann. Univ. Marie-Curie Sklodowska, **52** (7) (1998), 51–56.

[6] Miller, S. S. and Mocanu, P. T., *Differential Subordinations. Theory and Applications*, M. Dekker Inc., New York - Basel, 2000.

[7] Ozaki, S., *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku A, **40**(2) (1935), 167–188.

Received June 10, 2005

Faculty of Mathematics and Computer Science
"Babeş-Bolyai" University
Str. M. Kogălniceanu nr. 1
400084 Cluj-Napoca, Romania
E-mail: `bariczocsi@yahoo.com`