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EXISTENCE, UNIQUENESS AND DATA DEPENDENCE FOR THE SOLUTIONS OF VOLTERRA-FREDHOLM INTEGRAL EQUATIONS IN L^2 SPACES

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Abstract. In the present paper we give an existence and uniqueness theorem and a data dependence theorem for Volterra-Fredholm nonlinear integral equations, using the theory of weakly Picard operators.

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Key words. Fixed point, weakly Picard operator, data dependence, Volterra-Fredholm integral equation.

1. INTRODUCTION

In the present paper we consider the following nonlinear integral equations of mixed type:

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_a^b K(x,t,y,s,u(y,s))\mathrm{d}y\mathrm{d}s \quad (\lambda \in \mathbb{R}) \tag{1}$$

$$u(x,t) = f(x,t,u(a,0)) + \int_0^t \int_a^b K(x,t,y,s,u(y,s))\mathrm{d}y\mathrm{d}s \tag{2}$$

$\forall t \in [0,c]$, $\forall x \in [a,b]$. The Volterra-Fredholm type equations often arise from the mathematical modelling of the spreading, in space and time, of some contagious disease, in the theory of nonlinear parabolic boundary value problems and in many physical and biological models (for parabolic equations and Volterra-Fredholm integral equations, see: [1], [2], [3], [4]). Our aim is to give an existence and uniqueness theorem for (1) and then a data dependence theorem for (2), in $L^2([a,b]\times[0,c])$. Denote: $\overline{D} := [a,b]\times[0,c]$.

2. WEAKLY PICARD OPERATORS

Let $(X,\to)$ be an L-space and $A : X \to X$ an operator. In this paper we will use the following notations: $F_A := \{x \in X : A(x) = x\}$, $I(A) := \{Y \in P(X) : A(Y) \subset Y\}$, $A^0 := 1_X$, $A^{n+1} := A \circ A^n\ \forall n \in \mathbb{N}$. If (X,d) is a metric space, and $U,V \subset X$, then $H(U,V) := \max\left\{\sup_{u\in U}\inf_{v\in V} d(u,v), \sup_{v\in V}\inf_{u\in U} d(u,v)\right\}$ is the Pompeiu-Hausdorff functional.

Definition 2.1. (I. A. Rus [5]) The operator A is said to be:

(i) *weakly Picard operator* (*WPO*) if $\forall x_0 \in X\ A^n(x_0) \to x_0^*$, and the limit x_0^* is a fixed point of A, which may depend on x_0.

(ii) *Picard operator* (PO) if $F_A = \{x^*\}$ and $\forall x_0 \in X\ A^n(x_0) \to x^*$.

If A is an WPO, we consider the operator A^∞ defined by $A^\infty : X \to X$, $A^\infty(x) := \lim_{n\to\infty} A^n(x)$.

THEOREM 2.1. (I. A. Rus [5]) *The operator A is WPO if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ such that:*

(i) $X_\lambda \in I(A),\ \forall\lambda \in \Lambda$.

(ii) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is PO,* $\forall\lambda \in \Lambda$.

In order to study data dependence of the fixed points set of WPOs, a new class of operators was introduced, namely c-weakly Picard operators:

DEFINITION 2.2. (I. A. Rus) Let (X, d) a metric space and $c > 0$. An operator $A : X \to X$ is called *c-weakly Picard* (c-*WPO*) if it is a WPO and

$$d\big(x, A^\infty(x)\big) \le cd\big(x, A(x)\big), \quad \forall x \in X. \tag{3}$$

If A is PO and (3) is fulfilled, then A is said to be *c-Picard* (c-*PO*).

THEOREM 2.2. (I. A. Rus) *Let (X, d) be a complete metric space and $A, B : X \to X$ two operators. If*

(i) *there exist $c_1, c_2 > 0$ such that A is c_1-WPO and B is c_2-WPO;*

(ii) *there exists $\eta > 0$ such that $d\,(A(x), B(x)) \le \eta$, $\forall x \in X$,*

then $H(F_A, F_B) \le \eta \max\{c_1, c_2\}$.

3. MAIN RESULTS

In the beginning, consider the Volterra-Fredholm equation with parameter (1), for which we can give the following existence and uniqueness theorem based on the Contraction Principle:

THEOREM 3.1. *Assume that:*

(i) *$f \in L^2(\overline{D})$, $K : \overline{D} \times \overline{D} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions:*

a) *$K(\cdot,\cdot,\cdot,\cdot,u)$ is measurable on $\overline{D} \times \overline{D}$ for all $u \in \mathbb{R}$;*

b) *$K(x,t,y,s,\cdot)$ is continuous on $\mathbb{R}$ for almost any $(x,t,y,s) \in \overline{D} \times \overline{D}$;*

(ii) *there exists $n \in L^2(\overline{D})$ such that:*

$$\left|\int_0^t \int_a^b K(x,t,y,s,0)\mathrm{d}y\mathrm{d}s\right| \le n(x,t), \quad \forall (x,t) \in \overline{D}; \tag{4}$$

(iii) *there exists $L_K \in L^2(\overline{D} \times \overline{D})$ such that:*

$$|K(x,t,y,s,u) - K(x,t,y,s,v)| \le L_K(x,t,y,s)|u - v|$$

for all $(x,t), (y,s) \in \overline{D}$ and $u, v \in \mathbb{R}$;

(iv) $|\lambda| \cdot \|L_K\|_{L^2(\overline{D})} < 1$.

Then (1) has a unique solution $u^* \in L^2(\overline{D})$ and the sequence

$$u_{n+1}(x,t) = f(x,t) + \lambda \int_0^t \int_a^b K(x,t,y,s,u_n(y,s))\mathrm{d}y\mathrm{d}s \quad (n \in \mathbb{N})$$

converges to u^* for all $u_0 \in L^2(\overline{D})$.

Proof. Consider $A, \tilde{A} : L^2(\overline{D}) \to L^2(\overline{D})$ defined by:

$$\begin{aligned} A(u)(x,t) &:= f(x,t) + \lambda \int_0^t \int_a^b K(x,t,y,s,u(y,s))\mathrm{d}y\mathrm{d}s, \\ \tilde{A}(u)(x,t) &:= \int_0^t \int_a^b K(x,t,y,s,u(y,s))\mathrm{d}y\mathrm{d}s \end{aligned}$$

for all $u \in L^2(\overline{D})$. In virtue of (ii) and (iii), $\|\tilde{A}(u)\|_{L^2(\overline{D})} < +\infty$. But $f \in L^2(\overline{D})$, so the operator A is well defined. Now for all $u, v \in L^2(\overline{D})$ we have:

$$\|A(u) - A(v)\|_{L^2(\overline{D})} \leq |\lambda| \|L_K\|_{L^2(\overline{D}\times\overline{D})} \cdot \|u - v\|_{L^2(\overline{D})},$$

so A is an a-contraction with $a := |\lambda| \cdot \|L_K\|_{L^2(\overline{D})} < 1$. We can apply the Contraction Principle, so A is a Picard operator. □

In what follows we shall give a data dependence theorem for the Volterra-Fredholm equation of type (2). Consider the Volterra-Fredholm integral equations:

$$u(x,t) = f_1(x,t,u(a,0)) + \int_0^t \int_a^b K_1(x,t,y,s,u(y,s))\mathrm{d}y\mathrm{d}s \tag{5}$$

$$u(x,t) = f_2(x,t,u(a,0)) + \int_0^t \int_a^b K_2(x,t,y,s,u(y,s))\mathrm{d}y\mathrm{d}s \tag{6}$$

for all $(x,t) \in \overline{D} := [a,b] \times [0,c]$.

Theorem 3.2. *We suppose that:*

(i) *the functions $f_i : \overline{D} \times \mathbb{R} \to \mathbb{R}$ and $K_i : \overline{D} \times \overline{D} \times \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory conditions, for $i = 1, 2$;*

(ii) *there exist $m_1, m_2 \in L^2(\overline{D})$ and $n_1, n_2 \in L^2(\overline{D} \times \overline{D})$ such that:*

$$|f_i(x,t,u)| \leq m_i(x,t) \quad \forall (x,t) \in \overline{D}\ \forall u \in \mathbb{R}; \quad i = 1, 2; \tag{7}$$

$$\left| \int_0^t \int_a^b K_i(x,t,y,s,0)\mathrm{d}y\mathrm{d}s \right| \leq n_i(x,t) \quad \forall (x,t) \in \overline{D}; \quad i = 1, 2; \tag{8}$$

(iii) *there exist $L_{K_i} \in L^2(\overline{D} \times \overline{D})$, $i = 1, 2$ such that:*

$$|K_i(x,t,y,s,u) - K_i(x,t,y,s,v)| \leq L_{K_i}(x,t,y,s)|u - v| \tag{9}$$

for all $(x,t,y,s) \in \overline{D} \times \overline{D}$ and $u, v \in \mathbb{R}$, for $i = 1, 2$;

(iv) *$\|L_{K_i}\|_{L^2(\overline{D})} < 1$ for $i = 1, 2$;*

(v) *$f_i(a, 0, \lambda) = \lambda$, $\forall \lambda \in \mathbb{R}$, for $i = 1, 2$;*

(vi) *there exist η_1 and η_2 two positive constants such that $|f_1(x,t,u) - f_2(x,t,u)| \leq \eta_1$, $\forall (x,t,u) \in \overline{D} \times \mathbb{R}$ and $|K_1(x,t,y,s,u) - K_2(x,t,y,s,u)| \leq \eta_2$ for all $(x,t,y,s,u) \in \overline{D} \times \overline{D} \times \mathbb{R}$.*

If S_1, S_2 are the sets of solutions of equations (5) *and* (6) *in* $L^2(\overline{D})$*, then* $S_1 \neq \emptyset$*,* $S_2 \neq \emptyset$ *and*

$$H(S_1,S_2) \le \frac{\eta_1 + c(b-a)\eta_2}{1-\max\{\|L_{K_1}\|_{L^2(\overline{D})}, \|L_{K_2}\|_{L^2(\overline{D})}\}}.$$

Proof. We define the operators $A_1, A_2 : L^2(\overline{D}) \to L^2(\overline{D})$ by

$$A_i(u)(x,t) := f_i(x,t,u(a,0)) + \int_0^t \int_a^b K_i(x,t,y,s,u(y,s))\mathrm{d}y\mathrm{d}s$$

As in Theorem 3.1, conditions (i) and (ii) ensure that these operators are well defined. For all $\lambda \in \mathbb{R}$, let $X_\lambda := \{u \in L^2(\overline{D}) : u(a,0) = \lambda\}$. Then we have the partition $L^2(\overline{D}) = \bigcup\limits_{\lambda \in \mathbb{R}} X_\lambda$, the sets X_λ are closed and $X_\lambda \in I(A_i)$, $i = 1,2$. By Theorem 3.1, the restrictions $A_1|_{X_\lambda}$, respectively $A_2|_{X_\lambda}$ are contractions with constants $\|L_{K_1}\|_{L^2(\overline{D})}$, respectively $\|L_{K_2}\|_{L^2(\overline{D})}$. So, in virtue of Theorem 2.1, A_1 and A_2 are WPOs. Furthermore, A_i are c_i-WPOs with $c_i = \dfrac{1}{1-\|L_{K_i}\|_{L^2(\overline{D})}}$ for $i = 1,2$. From (vi) we have: $\|A_1(u) - A_2(u)\|_{L^2(\overline{D})} \le \eta_1 + c(b-a)\eta_2$. Now we can apply Theorem 2.2 and the conclusion follows. □

For other applications of weakly Picard operators theory in the study of data dependence of different equations solutions see [6], [7], [8], [9].

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