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SUFFICIENT CONDITIONS FOR STARLIKENESS AND CONVEXITY OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. Several interesting implications concerning analytic functions with negative coefficients are determined. In particular cases sufficient conditions for starlikeness, strongly starlikeness and convexity are obtained.

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1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions f, which are analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with normalization of the form $f(0) = f'(0) - 1 = 0$. R. Singh and S. Singh in [4] showed that for $f \in \mathcal{A}$ the following implication holds in U:

$$\operatorname{Re}[f'(z) + zf''(z)] > 0 \Rightarrow \operatorname{Re}\frac{zf'(z)}{f(z)} > 0. \tag{1}$$

P. T. Mocanu ([2], [3]) improved this result by

$$\operatorname{Re}\left[f'(z) + \frac{1}{2}zf''(z)\right] > 0 \Rightarrow \operatorname{Re}\frac{zf'(z)}{f(z)} > 0, \tag{2}$$

$$\operatorname{Re}[f'(z) + zf''(z)] > 0 \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{3} \tag{3}$$

and

$$\operatorname{Re}\left[f'(z) + \frac{1}{2}zf''(z)\right] > 0 \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{4\pi}{9}. \tag{4}$$

Other related results can be found in [1]. Let $\mathcal{N}$ denote the class of analytic functions with negative coefficients, that is

$$\mathcal{N} = \{f \in \mathcal{H}(U) \mid f(z) = z - \sum_{j=2}^{\infty} a_j z^j \;\; a_j \geq 0,\ j \geq 2\}.$$

In this paper we improve the above implications but in the particular case of analytic functions with negative coefficients.

2. PRELIMINARIES

We define the operator $D^n : \mathcal{N} \Rightarrow \mathcal{N},\ n \in \mathbb{N} = \{0, 1, 2, \cdots\}$ by

a) $D^0 f(z) = f(z)$;
b) $D^1 f(z) = Df(z) = zf'(z)$;
c) $D^n f(z) = D(D^{n-1} f(z)),\ z \in U$ (see [7]).

Let $S_n(\alpha)$, $n \in \mathbb{N}$, $\alpha \in [0, 1)$, be the class

$$S_n(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha,\ z \in U \right\}$$

and let $S_n[\beta]$, $n \in \mathbb{N}$, $\beta \in (0, 1]$, be the class

$$S_n[\beta] = \left\{ f \in \mathcal{A} : \left| \arg \frac{D^{n+1} f(z)}{D^n f(z)} \right| < \beta \frac{\pi}{2},\ z \in U \right\}.$$

We note that $S_0(0)$ is the class of starlike functions, $S_1(0)$ is the class of convex functions and $S_0[\beta]$ is the class of strongly starlike functions of order β; obviously $S_0[1] = S_0(0)$. We denote by $T_n(\alpha)$ the class $S_n(\alpha) \cap \mathcal{N}$.

For the functions in the classes $T_n(\alpha)$ ($n \in \mathbb{N}$) we have the next characterization theorem.

THEOREM A. *Let* $\alpha \in [0, 1)$, *let* $n \in \mathbb{N}$ *and let* f *be in* $\mathcal{N}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $(a_j \geq 0)$. *Then the next assertions are equivalent*

$$f \in T_n(\alpha); \tag{5}$$

$$\sum_{j=2}^{\infty} j^n (j - \alpha) a_j \leq 1 - \alpha; \tag{6}$$

$$\left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right| < 1 - \alpha,\ z \in U. \tag{7}$$

The result is sharp and the extremal functions are

$$f_j(z) = z - \frac{1 - \alpha}{j^n (j - \alpha)} z^j,\ j \in \{2, 3, \cdots\}.$$

A more general form of this theorem is proved in [8] and [9].

We note that $S_0(\alpha)$ and $S_1(\alpha)$ are the class of starlike functions of order α and the class of convex functions of order α, respectively. The classes $T_0(\alpha)$ and $T_1(\alpha)$ were introduced and studied by H. Silverman [5] (see also [6]).

REMARK A. Because (5) and (7) are equivalent we have that if $f \in T_n(\alpha)$, then $D^{n+1} f(z) / D^n f(z)$ belongs to the disc centered at 1 and having the radius $1 - \alpha$ and from this we deduce that, for $f \in \mathcal{N}$,

$$\mathrm{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha \Rightarrow \left| \arg \frac{D^{n+1} f(z)}{D^n f(z)} \right| < \arcsin(1 - \beta).$$

REMARK B. It is well known that $S_1(0) \subset S_0(1/2)$ (every convex function is starlike of order $1/2$); for the functions with negative coefficients and when $\alpha < 1$ it is proved that

$$\text{Re}\left[\frac{zf''(z)}{f'(z}+1\right] > \alpha \Rightarrow \text{Re}\frac{zf'(z)}{f(z)} > \frac{2}{3-\alpha} \tag{8}$$

and when $\alpha \in [0,1)$ we have (see [9])

$$T_1(\alpha) \subset T_0(2/(3-\alpha)). \tag{9}$$

3. MAIN RESULTS

THEOREM 1. *Let $n \in \mathbb{N}$, $\beta \in [-1,1)$, and let $f \in \mathcal{N}$.*

a) *If $\beta \in [0,1)$, then*

$$\text{Re}\frac{D^{n+2}f(z)}{z} > \beta \Rightarrow \text{Re}\frac{D^{n+2}f(z)}{D^{n+1}f(z)} > \beta \Leftrightarrow f \in T_{n+1}(\beta) \Rightarrow f \in T_n\left(\frac{2}{3-\beta}\right).$$

b) *If $\beta \in [-1,0)$, then*

$$\text{Re}\frac{D^{n+2}f(z)}{z} > \beta \Rightarrow \text{Re}\frac{D^{n+1}f(z)}{D^nf(z)} > \frac{2(1+\beta)}{3+\beta} \Leftrightarrow f \in T_n\left(\frac{2(1+\beta)}{3+\beta}\right).$$

c) *Furthermore, if $\beta \in (-1,0)$, then*

$$\text{Re}\frac{D^{n+2}f(z)}{z} > \beta \Rightarrow \text{Re}\frac{D^{n+2}f(z)}{D^{n+1}f(z)} > \frac{2\beta}{\beta+1}.$$

Proof. We have

$$\text{Re}\frac{D^{n+2}f(z)}{z} = 1 - \sum_{j=2}^{\infty} j^{n+2}a_jz^{j-1}$$

and, from $\text{Re}\left[D^{n+2}f(z)/z\right] > \beta$, by letting $z \to 1^-$, z real, we obtain

$$\sum_{j=2}^{\infty} j^{n+2}a_j \le 1-\beta.$$

From Theorem A we know that $f \in T_{n+1}(\alpha)$ if and only if

$$\sum_{j=2}^{\infty} j^{n+1}(j-\alpha)a_j \le 1-\alpha. \tag{10}$$

But (10) holds if $j^{n+1}(j-\alpha)/(1-\alpha) \le j^{n+2}/(1-\beta),\ j \in \{2,3,\cdots\}$, or

$$j(\alpha-\beta) \le \alpha(1-\beta),\ j \in \{2,3,\cdots\} \tag{11}$$

and this last inequality is true for $\alpha = \beta$; together Remark B this proves a).

Now we consider $\beta \in [-1,0)$ and we use that

$$\frac{j^n(j-\alpha)}{1-\alpha} \le \frac{j^{n+2}}{1-\beta},\ j \in \{2,3,\cdots\} \tag{12}$$

implies $f \in T_n(\alpha)$. But (12) is equivalent to

$$(1-\alpha)j^2 - (1-\beta)j + \alpha(1-\beta) \geq 0,\ j \in \{2, 3, \cdots\}.$$

These last inequalities hold if

$$\frac{1-\beta+\sqrt{(1-\beta)^2 - 4\alpha(1-\alpha)(1-\beta)}}{2(1-\alpha)} \leq 2 \tag{13}$$

and (13) is true for $\alpha \leq 2(1+\beta)/(3+\beta)$ and this gives that $f \in T_n(2(1+\beta)/(3+\beta))$.

In the case $\beta \in [-1, 0]$ the inequalities (11) are satisfied only if $\alpha - \beta \leq 0$; but it is sufficient that $2(\alpha-\beta) \leq \alpha(1-\beta)$ and this is equivalent to $\alpha \leq 2\beta/(1-\beta)$. This completes the proof of c).

Corollary 1.1. *If $f \in \mathcal{N}$, then*

$$\text{Re}[f'(z) + zf''(z)] > 0 \Rightarrow \text{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > 0 \Rightarrow Re\frac{zf'(z)}{f(z)} > \frac{2}{3},\ z \in U.$$

Proof. We put $n = 0$ in Theorem 1, a) and then we use (9).

Corollary 1.1 improves (1) for $f \in \mathcal{N}$.

Corollary 1.2. *If $f \in \mathcal{N}$ and $\beta \in [-1, 0)$, then*

$$\text{Re}[f'(z) + zf''(z)] > \beta \Rightarrow Re\frac{zf'(z)}{f(z)} > \frac{2(1+\beta)}{3+\beta},\ z \in U.$$

Proof. We put $n = 0$ in Theorem 1, b) and then we use (9).

We note that $\text{Re}[f'(z) + zf''(z)] > -1 \Rightarrow f \in T_0(0)$ (that is f is starlike). Corollary 1.2 is also an improvement of (1) for $f \in \mathcal{N}$.

Corollary 1.3. *Let $n \in \mathbb{N}$ and let $f \in \mathcal{N}$;*

a) *If $\beta \in [0, 1)$, then, for $z \in U$,*

$$\text{Re}\frac{D^{n+2}f(z)}{z} > \beta \Rightarrow \left|\arg\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right| < \arcsin(1-\beta)$$

and

$$\text{Re}\frac{D^{n+2}f(z)}{z} > \beta \Rightarrow \left|\arg\frac{D^{n+1}f(z)}{D^{n}f(z)}\right| < \arcsin\frac{1-\beta}{3-\beta};$$

b) *If $\beta \in [-1, 0)$, then, for $z \in U$,*

$$\text{Re}\frac{D^{n+2}f(z)}{z} > \beta \Rightarrow \left|\arg\frac{D^{n+1}f(z)}{D^{n}f(z)}\right| < \arcsin\frac{1-\beta}{3+\beta}.$$

Proof. Corollary 1.3 is a consequence of Theorem 1 and Remark A.

Theorem 2. *If $f \in \mathcal{N}$ satisfies*

$$\text{Re}[f'(z) + \gamma zf''(z)] > \beta,\ z \in U, \tag{14}$$

then

$$\text{Re}\left[\frac{zf''(z)}{f'(z)}+1\right]>\alpha,\ z\in U, \tag{15}$$

where

a) $\alpha=\alpha_1=\dfrac{2\beta+\gamma-1}{\gamma+\beta}$ *when* $\beta\in[-1,0]$ *and* $\gamma\geq 1$;

b) $\alpha=\alpha_2=\dfrac{\beta+\gamma-1}{\gamma}$ *when* $\beta\in[0,1)$ *and* $\gamma>0$.

Proof. We have

$$f'(z)+\gamma zf''(z)=1-\sum_{j=2}^{\infty}j[1+\gamma(j-1)]a_jz^{j-1}; \tag{16}$$

from (14), letting $z\to 1^-$ by real numbers, we obtain

$$\sum_{j=2}^{\infty}j[1+\gamma(j-1)]a_j\leq 1-\beta. \tag{17}$$

We note that (15) is implied by $\left|\frac{zf''(z)}{f'(z)}\right|<1-\alpha$.

But we have

$$|zf''(z)|-(1-\alpha)|f'(z)|=\left|\sum_{j=2}^{\infty}j(j-1)a_jz^{j-1}\right|-(1-\alpha)\left|1-\sum_{j=2}^{\infty}ja_jz^{j-1}\right|$$

$$\leq\sum_{j=2}^{\infty}j(j-1)a_j-(1-\alpha)\left(1-\sum_{j=2}^{\infty}ja_j\right)=\sum_{j=2}^{\infty}j(j-\alpha)a_j-(1-\alpha)$$

and we deduce that

$$\sum_{j=2}^{\infty}j[(j-\alpha)]a_j\leq 1-\alpha \tag{18}$$

implies (15).

The relation (18) holds if

$$\frac{j(j-\alpha)}{1-\alpha}\leq\frac{j[1+\gamma(j-1)]}{1-\beta},\ j\geq 2.$$

These last inequalities are equivalent to

$$j(1-\beta-\gamma+\alpha\gamma)\leq 1-\gamma+\alpha\gamma-\alpha\beta,\ j\geq 2 \tag{19}$$

and they can be satisfied only if $1-\beta-\gamma+\alpha\gamma\leq 0$.

If $\beta=0$ and $\alpha=\alpha_1$ (in this case $\alpha_1=\alpha_2$, too) (19) holds. If $\beta\in[-1,0)$, $\gamma\geq 1$ and $\alpha=\alpha_1$, then $1-\beta-\gamma+\alpha\gamma=\frac{\beta(1-\beta)}{\beta+\gamma}\leq 0$, and $1-\gamma+\alpha\gamma-\alpha\beta=$

$\frac{2\beta(1-\beta)}{\beta+\gamma} \le 0$. The inequalities (19) can be written

$$j \ge \frac{\alpha\beta - \alpha\gamma + \gamma - 1}{\beta + \gamma - \alpha\gamma - 1},\ j \ge 2 \tag{20}$$

and (20) holds because $\frac{\alpha_1\beta - \alpha_1\gamma + \gamma - 1}{\beta+\gamma-\alpha_1\gamma-1} = \frac{2\beta(1-\beta)}{\beta(1-\beta)} = 2$.

If $\beta \in [0,1]$, $\gamma > 0$ and $\alpha = \alpha_2$, then $1-\beta-\gamma+\alpha\gamma = 0$, $1-\gamma+\alpha\gamma-\alpha\beta = \beta(1-\beta)/\gamma \ge 0$ and (19) also holds.

From Theorem 2 we obtain the following sufficient conditions for convexity.

Corollary 2.1. *Let* $f \in \mathcal{N}$; *then* $\mathrm{Re}[f'(z) + \gamma z f''(z)] > \beta,\ z \in U \Rightarrow f \in T_1(\alpha)$, *where*

a) *if* $\beta \in [-1,0]$ *and* $\gamma \ge 1-2\beta$, *then* $\alpha = \alpha_1 = (2\beta+\gamma-1)/(\gamma+\beta)$;
b) *if* $\beta \in [0,1)$ *and* $\gamma \ge 1-\beta$, *then* $\alpha = \alpha_2 = (\beta+\gamma-1)/\gamma$.

From Theorem 2 we also obtain the following sufficient conditions for starlikeness.

Corollary 2.2. *Let* $f \in \mathcal{N}$; *then* $\mathrm{Re}[f'(z) + \gamma z f''(z)] > \beta,\ z \in U \Rightarrow f \in T_0(\delta)$, *where*

a) *if* $\beta \in [-1,0]$ *and* $\gamma > 1$, *then* $\delta = \delta_1 = \dfrac{2(\beta+\gamma)}{2\gamma+\beta+1} = 1 - \dfrac{1-\beta}{2\gamma+\beta+1}$
b) *if* $\beta \in [0,1)$ *and* $\gamma > 0$, *then* $\delta = \delta_2 = \dfrac{2\gamma}{2\gamma-\beta+1} = 1 - \dfrac{1-\beta}{2\gamma-\beta+1}$.

From Corollary 2.1 and 2.2, together with Remark A, we obtain

Corollary 2.3. *If* $f \in \mathcal{N}$; *then*

$$\mathrm{Re}[f'(z) + \gamma z f''(z)] > \beta,\ z \in U \Rightarrow \left|\arg\left(\frac{zf''(z)}{f'(z)} + 1\right)\right| < \delta,\ z \in U,$$

$$\mathrm{Re}[f'(z) + \gamma z f''(z)] > \beta,\ z \in U \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \lambda,\ z \in U,$$

where

a) $\delta = \dfrac{1-\beta}{\gamma+\beta}$ *when* $\beta \in [-1,0)$ *and* $\gamma > 1-2\beta$;

b) $\delta = \dfrac{1-\beta}{\gamma}$ *when* $\beta \in [0,1)$ *and* $\gamma > 1-\beta$;

c) $\lambda = \dfrac{1-\beta}{2\gamma+\beta+1}$ *when* $\beta \in [-1,0)$ *and* $\gamma > 1$;

d) $\lambda = \dfrac{1-\beta}{2\gamma-\beta+1}$ *when* $\beta \in [0,1)$ *and* $\gamma > 0$.

Theorem 3. *Let* $f \in \mathcal{N}$, $\beta \in [-1,1)$, $\gamma = -\beta$; *then*

$$\mathrm{Re}[f'(z) + \gamma z f''(z)] > \beta,\ z \in U \Rightarrow \mathrm{Re}\frac{zf'(z)}{f(z)} > 0,\ z \in U.$$

Proof. As in the proof of the Theorem 2, from $\mathrm{Re}[f'(z)+\gamma z f''(z)]>\beta$ we obtain (17) and by using Theorem A we have that $\mathrm{Re}[zf'(z)/f(z)]>0$ holds if

$$\sum_{j=2}^{\infty} j a_j \le 1. \tag{21}$$

Comparing (17) and (21) we find that (21) holds if

$$j \le \frac{j[1+\gamma(j-1)]}{1-\beta}.$$

But this is equivalent to $\beta+\gamma\ge 0$ and this completes the proof. □

4. INTEGRAL VERSIONS

Let $L_c:\mathcal{A}\to\mathcal{A}$ be the integral operator defined by $f=L_c(g)$, where $c\in(-1,\infty)$, $g\in\mathcal{A}$ and

$$f(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)\,\mathrm{d}t.$$

In the particular case $c=1$ we obtain the Libera integral operator L_1. A simple computation shows that

$$f'(z)+\frac{1}{c+1}zf''(z)=g'(z),\ z\in U$$

and because of this relationship Theorem 2 and Theorem 3 can be written in the following integral version.

THEOREM 2'. *If* $f=L_c(g)$, *where* $g\in\mathcal{N}$ *satisfies* $\mathrm{Re}g'(z)>\beta,\ z\in U$, *then* $\mathrm{Re}\,[zf''(z)/f'(z)+1]>\alpha,\ z\in U$, *where*

a) $\alpha=\alpha_1=\dfrac{2\beta+2\beta c-c}{1+\beta+\beta c}$ *when* $\beta\in[-1,0]$ *and* $c\in(-1,0]$;

b) $\alpha=\alpha_2=\beta+\beta c-c$, *when* $\beta\in[0,1)$ *and* $c\in(0,\infty)$.

THEOREM 3'. *Let* $g\in\mathcal{N}$, *let* $f=L_c(g)$, *where* $c=-(1+\beta)/\beta$ *and* $\beta\in[-1,1),\ \beta\neq 0$; *then* $\mathrm{Re}[g'(z)]>\beta,\ z\in U\Rightarrow \mathrm{Re}\frac{zf'(z)}{f(z)}>0,\ z\in U$.

5. PARTICULAR CASES

The next implications are consequences of Corollary 1.2, 1.4, 2.2, 2.3 or Theorem 3 and they improve (in the case of functions in $\mathcal{N}$) the implications (1), (2), (3) and (4), for $z\in U$.

$$\mathrm{Re}[f'(z)+zf''(z)]>-1\Rightarrow \mathrm{Re}\frac{zf'(z)}{f(z}>0$$

$$\mathrm{Re}[f'(z)+zf''(z)]>\frac{-1}{3}\Rightarrow \mathrm{Re}\frac{zf'(z)}{f(z}>\frac{1}{2}\Rightarrow\left|\arg\frac{zf'(z)}{f(z)}\right|<\arcsin(\frac{1}{2})=\frac{\pi}{6}$$

$$\mathrm{Re}[f'(z)+zf''(z)]>0 \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \arcsin(\frac{1}{3}) \simeq 19.47^\circ$$

$$\mathrm{Re}[f'(z)+zf''(z)]>0 \Rightarrow \mathrm{Re}\frac{zf'(z)}{f(z} > \frac{1}{2} \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \arcsin(\frac{1}{2}) = \frac{\pi}{6}$$

$$\mathrm{Re}[f'(z)+\frac{1}{2}zf''(z)] > \frac{-1}{2} \Rightarrow \mathrm{Re}\frac{zf'(z)}{f(z} > 0$$

$$\mathrm{Re}[f'(z)-\frac{1}{2}zf''(z)] > \frac{1}{2} \Rightarrow \mathrm{Re}\frac{zf'(z)}{f(z} > 0$$

$$\mathrm{Re}[f'(z)+\frac{1}{2}zf''(z)]>0 \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \arcsin(\frac{1}{2}) = \frac{\pi}{6}$$

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