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NONCONVEX MIXED QUASI VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce a new class of mixed quasi variational inequalities, known as nonconvex mixed quasi variational inequalities in the setting of g-convexity. We suggest some algorithms for solving nonconvex mixed quasi variational inequalities by using the auxiliary principle technique. The convergence of the proposed methods either requires partially relaxed strongly monotonicity or pseudomononicity. We also introduce the concept of well-posedness for the nonconvex mixed quasi variational inequalities. As special cases, we obtain a number of known and new results for solving various classes of equilibrium and variational inequality problems. Our results can be considered as a significant improvement of the previously known results.

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1. INTRODUCTION

Variational inequalities theory has witnessed an explosive growth in theoretic advances, algorithmic aspects and applications across all discipline of pure and applied sciences. This theory provides a novel and unified treatment of problems arising in economics, finance, transportation, network and structural analysis, elasticity and optimization. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative, see [1–25] and the references therein. Almost all the results obtained so far in this area are in the setting of convexity. It has been noted that these results may not hold in the nonconvex setting. In recent years, the concept of convexity has been generalized in many directions, which has potential and important applications in various fields . A significant generalization of the convex functions is the introduction of g-convex functions. It is well known that the g-functions and g-convex sets may not be convex functions and convex sets, see [3, 15, 24]. However, it can be shown that the class of g-convex function have some nice properties, which the convex functions have. In particular, it been shown [15] that the minimum of the g-functions over the g-convex sets can be characterized by a class of variational inequalities, which are called *nonconvex (g-convex) variational inequalities.* Inspired and motivated by the recent research work going in this field, we consider a new class of variational inequalities, which is called *nonconvex mixed quasi variational inequalities,* where the convex set is replaced by the so-called g-convex set. For $g = I$, the indentity operator, we obtain the original mixed quasi variational inequalities, which have been studied extensively in recent years, see [1, 2, 4–7, 18–20]. There are several numerical methods including projection and its variant forms, Wiener-Hopf equations, descent and auxiliary principle for solving variational inequalities. On the other hand, there are only few iterative methods for solving mixed quasi variational inequalities. It is known that projection methods and variant forms including Wiener-Hopf equations cannot be extended for mixed quasi variational inequalities involving the nonlinear (nondifferentiable) bifunctions. This fact has motivated to use the auxiliary principle technique, which is mainly due to Lions and Stampacchia [9]. Glowinski, Lions and Tremolieres [7] used this technique to study the existence of a solution of the mixed variational inequalities. In recent years, this technique has been used to suggest and analyze various iterative methods for solving various classes of variational inequalities. It has been shown that a substantial number of numerical methods can be obtained as special cases from this technique, see [16–18, 21, 25] and references therein. We again use the auxiliary principle to suggest a class of iterative methods for solving nonconvex mixed quasi variational inequalities. The convergence of these methods requires only that the operator is partially relaxed strongly monotone, which is weaker than monotonicity. Consequently, we improve the convergence results of previously known methods, which can be obtained as special cases from our results. We also use the auxiliary principle technique to suggest and analyze a proximal method for solving equilibrium problem, which was introduced in Martinet [12] as a regularization of convex optimization in Hilbert space. For the recent applications and developments of the proximal methods, see, for example, [16, 17, 21, 23]. We prove that the convergence of proximal method requires only pseudomonotonicity, which is a weaker condition. This clearly improves the known results. Since nonconvex mixed quasi variational inequalities includes (nonconvex) variational inequalities and complementarity problems as special cases, results obtained in this paper continue to hold for these problems. Our results can be considered an important and significant extension of the known results for solving variational inequalities and related optimization problems.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty and closed set in H.

First of all, we recall the following concepts and results.

Definition 1. Let K be any set in H. The set K is said to be g-convex, if there exists a function $g : K \longrightarrow K$ such that

$$g(u) + t(g(v) - g(u)) \in K, \quad \forall \quad u, v \in K, t \in [0, 1].$$

Note that every convex set is g-convex, but the converse is not true, see [3, 24].

From now onward, we assume that K is a g-convex set, unless otherwise specified.

DEFINITION 2. The function $f : K \longrightarrow H$ is said to be g-convex, if

$$\begin{aligned} f(g(u)+t(g(v)-g(u))) &\leq (1-t)f(g(u)) \\ &\quad +tf(g(v)). \quad \forall u,v \in K, t \in [0,1]. \end{aligned}$$

Clearly every convex function is g-convex, but the converse is not true, see [24].

DEFINITION 3. A function f is said to be strongly g-convex on the g-convex set K with modulus $\mu > 0$, if, $\forall u, v \in K, t \in [0,1]$,

$$\begin{aligned} f(g(u)+t(g(v)-g(u)) &\leq (1-t)f(g(u))+tf(g(v)) \\ &\quad -t(1-t)\mu\|g(v)-g(u)\|^2. \end{aligned}$$

Using the convex analysis techniques, one can easily show that the differentiable g-convex function F is strongly g-convex function if and only if

$$f(g(v)) - f(g(u)) \geq \langle f'(g(u)), g(v)-g(u)\rangle + \mu\|g(v)-g(u)\|^2$$

or

$$\langle f'(g(u)) - f'(g(v)), g(u)-g(v)\rangle \geq 2\mu\|g(v)-g(u)\|^2,$$

that is, $f'(g(u))$ is a strongly monotone operator.

It is well known [3, 24] that the g-convex functions are not convex function, but they have some nice properties which the convex functions have. Note that for $g = I$, the g-convex functions are convex functions and definition 3 is a well known result in convex analysis.

For a given nonlinear continuous operator $T : K \longrightarrow H$, consider the problem of finding $u \in K$ such that

$$(1) \quad \langle Tg(u), g(v)-g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall \in K,$$

where $\varphi(.,) : H \times H \longrightarrow R \cup \{\infty\}$ is a continuous bifunction in both variables. Inequality (1) is known as the *nonconvex mixed quasi variational inequality.* It is worth mentioning that nonconvex mixed quasi variational inequalities (1) are quite different from the so-called general mixed quasi variational inequalities. For the applications and numerical methods of general mixed quasi variational inequalities, see [18, 19] and the references therein.

If $g = I$, the indentity operator, then the g-convex set K becomes the convex set K, and consequently the nonconvex mixed quasi variational inequalities (1) are equivalent to finding $u \in K$ such that

$$(2) \qquad \langle Tu, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall \quad v \in K,$$

which are known as the mixed quasi variational inequalities. Mixed quasi variational inequalities have been studied extensively in recent years, see [1, 2, 4–8, 13, 14, 18–20].

We remark that if $K(u)$ is a closed convex-valued set in H and

$$\phi(u,u) = \begin{cases} 0, & \text{if } u \in K(u) \\ +\infty, & \text{otherwise,} \end{cases}$$

is the indicator function of K, then the problem (3) is equivalent to finding $u \in K(u)$

$$\langle T(g(u)), g(v) - g(u)\rangle \geq 0, \quad \forall v \in K(u), \tag{3}$$

which is called the nonconvex quasi general variational inequality. In particular, if $K(u) \equiv K$, the convex set and $g = I$, the identity operator, then one can obtain the original variational inequality, that is, find $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K, \tag{4}$$

introduced and studied by Stampacchia [22] in 1964. In brief, for a suitable and appropriate choice of the operators T, g, and the space H, one can obtain a wide class of variational inequalities and complementarity problems. This clearly shows that problem (1) is quite general and unifying one. Furthermore, problem (1) has important applications in various branches of pure and applied sciences, see [1–25].

We also need the following concepts.

Definition 4. The function $T : K \to H$ is said to be:

(i) *partially relaxed strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle T(g(u)) - T(g(v)), g(z) - g(v)\rangle \geq \alpha \|g(z) - g(u)\|^2, \quad \forall u, v, z \in K.$$

(ii) *monotone*, if

$$\langle T(g(u)) - T(g(v)), g(u) - g(v)\rangle \geq 0, \quad \forall u, v \in K.$$

(iii) *pseudomonotone*, if

$$\begin{aligned} &\langle T(g(u)), g(v) - g(u)\rangle + \varphi(v,u) - \varphi(u,u) \geq 0 \\ &\Longrightarrow \\ &\langle T(g(v)), g(u) - g(v)\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall u, v \in K. \end{aligned}$$

(iv) *hemicontinuous,* if $\forall u, v \in K, t \in [0,1]$, the mapping

$$\langle T(g(u) + t(g(v) - g(u)), g(v) - g(u))\rangle$$

is continuous.

We remark that if $z = u$, then partially relaxed strongly monotonicity is exactly monotonicity of the operator T. For $g \equiv I$, the indentity operator, then Definition 4 reduces to the standard definition of partially relaxed strongly monotonicity, monotonicity, and pseudomonotonicity. It is known that monotonicity implies pseudomonotonicity, but not conversely. This implies that the concepts of partially relaxed strongly monotonicity and pseudomonotonicity are weaker than monotonicity.

LEMMA 1. *Let T be pseudomonotone, hemicontinuous and the bifunction $\varphi(.,.)$ is g-convex with respect to second argument. Then the nonconvex mixed quasi variational inequality* (1) *is equivalent to finding $u \in K$ such that*

$$\langle T(g(v)), g(v) - g(u)\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in K. \tag{5}$$

Proof. Let $u \in K$ be a solution of (1). Then

$$\langle T(g(u)), g(v) - g(u)\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in K$$

which implies

$$\langle T(g(v)), g(v) - g(u)\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in K,$$

since T is pseudomonotone.

Conversely, let $u \in K$ satisfy (5). Since K is a g-convex set, $\forall u, v \in K, t \in [0,1], g(v_t) = g(u) + t(g(v) - g(u)) \equiv (1-t)g(u) + tg(v) \in K$.

Taking $g(v) = g(v_t)$ in (5), we have

$$\begin{aligned} t\langle T(g(v_t)), g(u) - g(v)\rangle &\leq \varphi(g(v_t), g(u)) - \varphi(g(u)), g(u)) \\ &\leq t\{\varphi(g(v), g(u) - \varphi(g(u), g(u))\}, \end{aligned} \tag{6}$$

where we have used the fact that the bifunction $\varphi(.,.)$ is g-convex with respect to the second argument.

Dividing the inequality (6) by and taking the limit as $t \longrightarrow 0$, we have

$$\langle T(g(u)), g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u).g(u)) \geq 0, \forall v \in K,$$

the required result. □

REMARK 1. Problem (5) is known as the dual nonconvex mixed quasi variational inequality. Lemma 1 can be viewed as a generalization and an extension of Minty's Lemma.

3. ITERATIVE SCHEMES

In this section, we suggest and analyze some new iterative methods for solving the problem (1) by using the auxiliary principle technique of Glowinski, Lions and Tremolieres [7] as developed by Noor [16–18] in recent years.

For a given $u \in K$, consider the problem of finding a unique $w \in K$ satisfying the auxiliary variational inequality

$$\begin{aligned} \langle \rho T(g(u)) \quad + \quad & E'(g(w)) - E'(g(u)), g(v) - g(w)\rangle \\ & +\rho\varphi(g(v), g(w)) - \rho\varphi(g(w), g(w)) \geq 0, \quad \forall v \in K, \end{aligned} \tag{7}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly g-convex function E at $u \in K$. Problem (7) has a unique solution, since the function E is strongly g-convex function.

REMARK 2. The function $B(w,u) = E(g(w)) - E(g(u)) - \langle E'(g(u)), g(w) - g(u)\rangle$ associated with the g-convex function $E(u)$ is called the generalized Bregman function. We note that if $g = I$, then $B(w,u) = E(w) - E(u) - \langle E'(u), w - u\rangle$ is the well known Bregman function. For the applications of the Bregman function in solving variational inequalities and complementarity problems, see[16, 25] and the references therein.

We note that if $w = u$, then clearly w is a solution of the nonconvex mixed quasi variational inequality (1). This observation enables us to suggest the following method for solving the nonconvex mixed quasi variational inequalities (1).

ALGORITHM 1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} \langle \rho T(g(u_n)) \quad &+ \quad E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1})\rangle \\ &+ \quad \rho\varphi(g(v), g(u_{n+1})) - \rho\varphi(g(u_{n+1}), g(u_{n+1})) \geq 0, \ \forall v \in K, \end{aligned} \tag{8}$$

where $\rho > 0$ is a constant.

Note that if $g \equiv I$, the identity operator, the g-convex set K becomes a convex set K, then Algorithm 1 reduces to a method for solving the mixed quasi variational inequalities (2).

ALGORITHM 2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} \langle \rho T(u_n) \quad &+ \quad E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \\ &+ \quad \varphi(v, u_{n+1}) - \varphi(u_{n+1}, u_{n+1}) \geq 0, \quad \forall v \in K, \end{aligned}$$

which appears to be a new one.

For suitable and appropriate choice of the operators and the space H, one can obtain various new and known methods for solving equilibrium, variational inequalities and complementarity problems.

For the convergence analysis of Algorithm 1, we need the following result. The analysis is in the spirit of Noor [17].

THEOREM 1. *Let $E(u)$ be a strongly g-convex with modulus $\beta > 0$. and the operator T is partially relaxed strongly monotone with constant $\alpha > 0$. If the bifunction $\varphi(.,.)$ is skew-symmetric and $0 < \rho < \frac{\beta}{\alpha}$, then the approximate solution u_{n+1} obtained from Algorithm 1 converges to a solution of (1).*

Proof. Let $u \in K$ be a solution of (1). Then

$$\langle Tg(u), g(v) - g(u)\rangle + \varphi(g(v), g(u)) \geq \varphi(g(u), g(u)) \geq 0, \ \forall v \in K. \tag{9}$$

Now taking $v = u_{n+1}$ in (9) and $v = u$ in (8), we have

$$\langle T(g(u)), g(u_{n+1}) - g(u)\rangle + \varphi(g(u_{n+1}), g(u)) \geq \varphi(g(u), g(u)). \tag{10}$$

and

$$\begin{aligned}\langle \rho T(g(u_n)) \quad + \quad & E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \\ & +\rho\{\varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(u), g(u_{n+1})) \geq 0.\end{aligned} \tag{11}$$

We now consider the function

$$\begin{aligned} B(u,w) &= E(g(u)) - E(g(w)) - \langle E'(g(w)), g(u) - g(w)\rangle \\ &\geq \beta\|g(u) - g(w)\|^2, \end{aligned} \tag{12}$$

using strongly g-convexity of E. Now combining (10), (11) and (12), we have

$$\begin{aligned} B(u, u_n) - B(u, u_{n+1}) &= E(g(u_{n+1})) - E(g(u_n)) \\ &\quad -\langle E'(g(u_n)), g(u_{n+1}) - g(u_n) \\ &\quad +\langle E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \\ &\geq \beta\|g(u_{n+1}) - g(u_n)\|^2 \\ &\quad +\langle E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \\ &\geq \beta\|g(u_{n+1}) - g(u_n)\|^2 \\ &\quad +\rho\langle T(g(u_n)) - T(g(u)), g(u_{n+1} - g(u))\rangle \\ &\quad +\rho\{\varphi(g(u), g(u)) - \varphi(g(u), g(u_{n+1})) \\ &\quad -\varphi(g(u_{n+1}), g(u)) + \varphi(g(u_{n+1}), g(u_{n+1}))\} \\ &\geq \{\beta - \rho\alpha\}\|g(u_{n+1}) - g(u_n)\|^2 \\ &\quad +\rho\{\varphi(g(u), g(u)) - \varphi(g(u), g(u_{n+1})) \\ &\quad -\varphi(g(u_{n+1}), g(u)) + \varphi(g(u_{n+1}), g(u_{n+1}))\} \\ &\geq \{\beta - \rho\alpha\}\|g(u_{n+1}) - g(u_n)\|^2, \end{aligned}$$

where we have used the fact that T is partially relaxed strongly monotone with constant $\alpha > 0$ and the bifunction $\varphi(.,.)$ is skew-symmetric.

If $u_{n+1} = u_n$, then clearly u_n is a solution of the nonconvex problem (2.1). Otherwise, for $0 < \rho < \frac{\beta}{\alpha}$, it follows that $B(u, u_n) - B(u, u_{n+1})$ is nonnegative, and we must have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

Now using the technique of Zhu and Marcotte [25], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the nonconvex mixed quasi variational inequalities (1). $\square$

We now show that the auxiliary principle technique can be used to suggest and analyze a proximal method for solving nonconvex mixed quasi variational inequalities (1). We show that the convergence of the proximal method requires only the pseudomonotonicity, which is a weaker condition than monotonicity.

For a given $u \in K$ consider the auxiliary problem of finding a unique $w \in K$ such that

$$\begin{aligned} &\langle \rho T(g(w)) + E'(g(w)) - E'(g(u)), g(v) - g(w)\rangle \\ &+\rho\{\varphi(g(v), g(w)) - \varphi(g(w), g(w))\} \geq 0,\ \forall \in K, \end{aligned} \tag{13}$$

where $\rho > 0$ is a constant. Note that if $w = u$, then w is a solution of (1). This fact enables us to suggest the following iterative method for solving nonconvex mixed quasi variational inequalities (1).

Algorithm 3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho T(g(u_{n+1})) + E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1})\rangle \\ &+\rho\{\varphi(g(v), g(u_{n+1}) - \varphi(g(u_{n+1}), g(u_{n+1}))\} \geq 0,\ \forall v \in K. \end{aligned} \tag{14}$$

Algorithm 3 is known as the proximal method for solving nonconvex problem (1). For $g \equiv I$, where I is the identity operator, the g-convex set K becomes the convex set K and we obtain a proximal method for the mixed quasi variational inequality (2), that is,

Algorithm 4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho T(u_{n+1}) + E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \\ &+\rho\{\varphi(v, u_{n+1}) - \varphi(u_{n+1}, u_{n+1}) \geq 0, \quad \forall v \in K, \end{aligned}$$

which appears to be a new one. Note that $E'(u)$ is the differential of a differentiable strongly convex function E at $u \in K$.

In a similar way, one can obtain a variant form of proximal methods for solving variational inequalities and complementarity problems as special cases.

We now study the convergence analysis of Algorithm 3 using the technique of Theorem 1. For the sake of completeness and to convey an idea of the techniques involved, we sketch the main points only.

Theorem 2. *Let $E(u)$ be a strongly g-convex with modulus $\beta > 0$. and the operator T be pseudomonotone. If the bifunction $\varphi(.,.)$ is skew-symmetric, then the approximate solution u_{n+1}obtained from Algorithm* 3 *converges to a solution of* (1).

Proof. Let $u \in K$ be a solution of (1). Then

$$\langle T(g(u)), g(v) - g(u)\rangle + \varphi(g(v), g(u)) \geq \varphi(g(u), g(u)), \forall v \in K,$$

which implies that

$$\langle T(g(v)), g(v) - g(u)\rangle + \varphi(g(v), g(u)) \geq \varphi(g(u), g(u)), \forall v \in K, \tag{15}$$

since T is pseudomonotone.

Taking $v = u_{n+1}$ in (15), we have

$$(16)\quad \langle T(g(u_{n+1})), g(u_{n+1}) - g(u)\rangle + \varphi(g(u_{n+1}), g(u)) \geq \varphi(g(u), g(u)).$$

Now as in Theorem 1, we have

$$\begin{aligned}
B(u,u_n) - B(u,u_{n+1}) &= E(g(u_{n+1})) - E(g(u_n)) \\
&\quad -\langle E'(g(u_n)), g(u_{n+1}) - g(u_n \\
&\quad +\langle E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \\
&\geq \beta\|g(u_{n+1}) - g(u_n)\|^2 \\
&\quad +\langle E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \\
&\geq \beta\|g(u_{n+1}) - g(u_n)\|^2 + \rho\{\varphi(g(u), g(u)) \\
&\quad -\varphi(g(u), g(u_{n+}) - \varphi(g(u_{n+1}), g(u)) \\
&\quad +\varphi(g(u_{n+1}), g(u_{n+1}))\} \\
&\geq \beta\|g(u_{n+1}) - g(u_n)\|^2,
\end{aligned}$$

using (16) and the fact that the bifunction $\varphi(.,.)$ is skew-symmetric.

If $u_{n+1} = u_n$, then clearly u_n is a solution of the nonconvex problem (1). Otherwise, it follows that $B(u,u_n) - B(u,u_{n+1})$ is nonnegative , and we must have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

Now using the technique of Zhu and Marcotte [25], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the nonconvex mixed quasi variational inequality (1). $\square$

4. WELL-POSEDNESS

In recent years, much attention has been given to introduce the concept of well-posedness for variational inequalities, see [8, 10, 11, 16] and the references therein. In this Section, we introduce the similar concepts of well-posedness for nonconvex mixed quasi variational inequalities (1). The results obtained can be considered as a natural generalization of previous results of [8, 10, 11, 16]. For this purpose, we need the following concepts.

For a given $\epsilon > 0$, we consider the sets

$$\begin{aligned}
A(\epsilon) = \{u \in K : \langle Tg(u), g(v) - g(u)\rangle &+ \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \\
&\geq -\epsilon\|g(v) - g(u)\|, \quad \forall v \in K\}
\end{aligned}$$

and

$$\begin{aligned}
B(\epsilon) = \{u \in K : \langle Tg(v), g(v) - g(u)\rangle &+ \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \\
&\geq -\epsilon\|g(v) - g(u)\|, \quad \forall v \in K\}.
\end{aligned}$$

For a nonempty set $X \subset H$, we define the diameter of X, denoted by $D(X)$, as

$$D(X) = \sup\{\|v-u\|; \quad \forall u, v \in X\}$$

Definition 5. We say that the nonconvex mixed quasi variational inequality (1) is *well-posed,* if and only if

$$A(\epsilon) \neq \phi \quad \text{and} \quad D(A(\epsilon)) \longrightarrow 0, \text{as} \quad \epsilon \longrightarrow 0.$$

For $g = I$, we obtain the correspeonding definition of well-posedness for mixed quasi variational inequalities (2), which is a natural extension of the one considered by Lucchetti and Patrone [10, 11] for variational inequalities.

Theorem 3. *Let the operator T be pseudomonotone hemicontinuous and let the bifunction $\varphi(.,.)$ be g-convex in the second argument. Then*

$$A(\epsilon) = B(\epsilon).$$

Proof. Let $u \in K$ be such that

$$\begin{aligned}\langle Tg(u), g(v)-g(u)\rangle \quad &+ \quad \varphi(g(v),g(u)) - \varphi(g(u),g(u)) \\ &\geq -\epsilon\|g(v)-g(u)\|, \quad \forall v \in K,\end{aligned}$$

which implies that

$$\begin{aligned}\langle Tg(v), g(v)-g(u)\rangle \quad &+ \quad \varphi(g(v),g(u)) - \varphi(g(u),g(u)) \\ &\geq -\epsilon\|g(v)-g(u)\|, \quad \forall v \in K,\end{aligned} \tag{17}$$

since T is pseudomonotone.

Thus

$$A(\epsilon) \subset B(\epsilon). \tag{18}$$

Conversely, let $u \in K$ such that (17) hold. Since K is a g-convex set, $\forall u, v \in K, t \in [0,1], \quad g(v_t) = g(u) + t(g(v)-g(u)) \equiv (1-t)g(u) + tg(v) \in K$.

Taking $g(v) = g(v_t)$ in (17)and using the g-convexity of $\varphi(.,.)$ in the second argument, we have

$$\begin{aligned}t\{\langle Tg(v_t), g(v)-g(u)\rangle \quad &+ \quad \varphi(g(v),g(u)) - \varphi(g(u),g(u))\} \\ &\geq -t\epsilon\|g(v)-g(u)\|.\end{aligned}$$

Dividing the above inequality by t and letting $t \longrightarrow 0$, we have

$$\langle Tg(u), g(v)-g(u)\rangle + \varphi(g(v),g(u)) - \varphi(g(u),g(u)) \geq -\epsilon\|g(v)-g(u)\|$$

which implies that

$$B(\epsilon) \subset A(\epsilon). \tag{19}$$

Thus from (17) and (19), we have

$$A(\epsilon) = B(\epsilon),$$

the required result. □

THEOREM 4. *If the bifunction is lower semicontinuous, that is,*

$$\varphi(u_n, u_n) \longrightarrow \varphi(u, u)$$

as $n \longrightarrow n$, *then the set* $B(\epsilon)$ *is closed under the assumptions of Theorem* 3.

Proof. Let $\{u_n : n \in N\} \subset B(\epsilon)$ be such that $g(u_n) \longrightarrow g(u)$ in K as $n \longrightarrow \infty$. This implies that $g(u_n) \in K$ and

$$\begin{aligned}\langle Tg(v), g(v) - g(u_n)\rangle \quad + \quad & \varphi(g(v), g(u_n)) - \varphi(g(u_n), g(u_n)) \\ & \geq -\epsilon\|g(v) - g(u_n)\|, \quad \forall v \in K.\end{aligned}$$

Taking the limit in the above inequality as $n \longrightarrow \infty$, we have

$$\begin{aligned}\langle Tg(v), g(v) - g(u)\rangle \quad + \quad & \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \\ & \geq -\epsilon\|g(v) - g(u)\|, \quad \forall v \in K,\end{aligned}$$

which implies that $u \in K$, since K is a closed and g-convex set. Consequently, it follows that the set $B(\epsilon)$ is closed. □

Using essentially the technique of Goeleven and Mantague [8], we can prove the following results. To convey an idea and for the sake of completeness, we include their proofs.

THEOREM 5. *Let* T *be pseudomonotone and hemicontinuous. If* $\varphi(.,.)$ *is* g*-convex in the second argument and the problem* (1) *is well-posed, then the nonconvex mixed quasi variational inequality* (1) *has a unique solution*

Proof. Let us define the sequence $\{u_k : k \in N\}$ by $g(u_k) \in A(1/k)$. Let $\epsilon > 0$ be sufficiently small and let $m, n \in N$ such that $n \geq m \geq \frac{1}{\epsilon}$. Then

$$A\left(\frac{1}{n}\right) \subset A\left(\frac{1}{m}\right) \subset A(\epsilon).$$

Thus

$$\|u_n - u_m\| \leq D\left(A\left(\frac{1}{n}\right)\right),$$

which implies that the sequence $\{u_n\}$ is a Cauchy sequence and it converges, that is, $u_k \longrightarrow u$ in K. From Theorem 3 and Theorem 4, we know that the set $A(\epsilon)$ is a closed set. Thus

$$u \in \cup_{\epsilon>0} A(\epsilon),$$

so that u is solution of the nonconvex problem (1). From the second condition of well-posedness, we see that the solution of the nonconvex mixed quasi variational inequality (1) is unique. □

THEOREM 6. *Let all the assumptions of Theorem* 3 *hold. If* $A(\epsilon) \neq 0, \forall \epsilon > 0$, $A(\epsilon)$ *is bounded for some* ϵ_0, *and* g *is a Lipschitz continuous function with constant* $\mu > 0$, *then the nonconvex mixed quasi variational inequality* (1) *has at least one solution.*

Proof. Let $u_n \in A(1/n)$. Then $A(1/n) \subset A(\epsilon)$, for n large enough. Thus for some subsequence $u_n \longrightarrow u \in K$, we have

$$\begin{aligned}\langle Tg(v), g(v) - g(u_n)\rangle \quad + \quad & \varphi(g(v), g(u_n)) \\ & \geq \varphi(g(u_n), g(u_n)) - \frac{1}{n}\|g(v) - g(u_n)\|, \quad \forall v \in K \\ \geq \quad & \varphi(g(u_n), g(u_n)) - \frac{\mu}{n}\{\|v\| + c\}, \quad \forall v \in K.\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we have

$$\langle Tg(v), g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0,$$

which implies that $g(u) \in B(0) = A(0)$, by Theorem 3. This shows that $g(u) \in A(0)$, from which it follows that the nonconvex problem (1) has at least one solution. □

Remark 3. **I.** If the nonconvex problem (1) has a unique solution, then it is clear that $A(\epsilon) \neq 0, \forall \epsilon > 0$ and $\cap_{\epsilon > 0} A(\epsilon) = \{u_0\}$.

II. It is known that [11] if the variational inequality (4) has a unique solution, then it is not well-posed.

III. From Theorem 5, we conclude that the unique solution of the nonconvex problem (1) can be computed by using the nonconvex mixed quasi ϵ-variational inequality, that is, find $u_\epsilon \in K$ such that

$$\begin{aligned}\langle Tg(u_\epsilon) \quad , \quad & g(v) - g(u_\epsilon)\rangle + \varphi(g(v), g(u_\epsilon)) \\ & -\varphi(g(u_\epsilon), g(u_\epsilon)) \geq -\epsilon\|g(v) - g(u_\epsilon)\|, \quad v \in K.\end{aligned}$$

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