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ON δ-PREOPEN SETS

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Abstract. The aim of this paper is to investigate some properties of the class of δ-preopen sets as the weaker form of open sets.

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1. PRELIMINARIES

It is well known that a large number of papers is devoted to the study of classes of subsets of a topological space, containing the class of open sets, and possessing properties more or less similar to those of open sets. Various notions of weaker form of open sets have been introduced in the literature, for example α-open sets [7], semiopen sets [4], preopen sets [5], β-open sets [3], semipreopen sets [1]. One of them is the notion of δ-preopen sets. The notion of a generalized class of open sets, called δ-preopen sets was introduced by Raychaudhuri and Mukherjee [9] in 1993. The aim of this paper investigate various properties of δ-preopen sets.

Throughout this paper, spaces X and Y mean topological spaces. Let A be a subset of a space X. For a subset A of (X, τ), $cl(A)$ and $int(A)$ represent the closure of A and the interior of A, respectively.

A subset A of a space X is called preopen [5] if $A \subset int(cl(A))$. The intersection (resp. union) of all preclosed (resp. preopen) sets containing (contained in) a set A is called the preclosure (resp. preinterior) of A and is denoted by $pcl(A)$ (resp. $pint(A)$) [2, 5].

A subset A of a space X is said to be regular open (respectively regular closed) if $A = int(cl(A))$ (respectively $A = cl(int(A))$) [10].

The δ-interior [11] of a subset A of X is the union of all regular open sets of X contained in A is denoted by $\delta\text{-}int(A)$. A subset A is called δ-open [11] if $A = \delta\text{-}int(A)$, i. e., a set is δ-open if it is the union of regular open sets.

The complement of δ-open set is called δ-closed. A set A of (X, τ) is called δ-closed [11] if $A = \delta\text{-}cl(A)$, where $\delta\text{-}cl(A) = \{x \in X : A \cap int(cl(U)) \neq \emptyset,$ $U \in \tau$ and $x \in U\}$.

2. PROPERTIES OF δ-PREOPEN SETS

Definition 1. A subset S of a topological space X is said to be δ-preopen [9] iff $S \subset int(\delta\text{-}cl(S))$. The complement of a δ-preopen set is called a δ-preclosed set [9].

Definition 2. The union (resp. intersection) of all δ-preopen (resp. δ-preclosed) sets, each contained in (resp. containing) a set S in a topological space X is called the δ-preinterior (resp. δ-preclosure) of S and it is denoted by δ-$pint(S)$ (resp. δ-$pcl(S)$) [9].

Proposition 1. *For any subset S of a topological space X, $X \backslash \delta$-$pcl(K) = \delta$-$pint(X \backslash K)$* [8].

The family of all δ-preopen (resp. δ-preclosed) sets of X is denoted by $\delta PO(X)$ (resp. $\delta PC(X)$). The family of all δ-preopen sets of X containing a point $x \in X$ is denoted by $\delta PO(X, x)$.

Proposition 2. *A union of any collection of δ-preopen sets of a topological space X is a δ-preopen set and a intersection of any collection of δ-preclosed sets of X is a δ-preclosed set* [9].

Proposition 3. *Let X be a topological space and $K \subset X$ and $M \subset X$. The following properties hold:* (1) *K is δ-preclosed if and only if $K = \delta$-$pcl(K)$,* (2) *If $K \subset M$, then δ-$pcl(K) \subset \delta$-$pcl(M)$,* (3) *δ-$pcl(\delta$-$pcl(K)) = \delta$-$pcl(K)$,* (4) *δ-$pcl(K)$ is δ-preclosed in X* [9].

Definition 3. Let X be a topological space and $x \in X$. A subset U of X is called a δ-preneighbourhood of x if and only if there exists a $S \in \delta PO(X)$ such that $x \in S \subset U$ [9].

Theorem 1. *Let X be a topological space and $S \subset X$. S is δ-preopen if and only if it is a δ-preneighbourhood of each its points.*

Proof. Let $x \in S$. Since $x \in S \subset S$, then S is a δ-preneighbourhood of each of its points.

Conversely, let $x \in S$. Since S is a δ-preneighbourhoodof each of its points, there exists $G_x \in \delta PO(X)$ such that $G_x \subset S$. We have $S = \bigcup_{x \in S} G_x$. Thus, S is a δ-preopen set. □

Definition 4. Let S be a subset of a topological space X and $x \in X$. x is called a δ-preinterior point of S if and only if $x \in \delta$-$pint(S)$.

Theorem 2. *Let X be a topological space and $S \subset X$ and $x \in X$. x is a δ-preinterior point of S if and only if S is a δ-preneighbourhood of x.*

Theorem 3. *Let K and S be subsets of a topological space X. Then*
(1) *δ-$pint(K) \cup \delta$-$pint(S) \subset \delta$-$pint(K \cup S)$,*
(2) *δ-$pint(K \cap S) \subset \delta$-$pint(K) \cap \delta$-$pint(S)$,*
(3) *δ-$pcl(K) \cup \delta$-$pcl(S) \subset \delta$-$pcl(K \cup S)$,*
(4) *δ-$pcl(K \cap S) \subset \delta$-$pcl(K) \cap \delta$-$pcl(S)$.*

Remark 1. The following examples show that inclusion can not be replaced with equality.

Example 1. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a, d\}, \{c\}, \{a, c, d\}\}$. Consider the sets $K = \{a, b, c\}$ and $S = \{b, c, d\}$. Then we obtain δ-$pint(K) \cap \delta$-$pint(S) = \{a, b, c\} \cap \{b, c, d\} = \{b, c\} \nsubseteq \delta$-$pint(K \cap S) = \{c\}$. If we take the sets $K = \{b, c\}$ and $S = \{d\}$, then we obtain δ-$pint(K \cup S) = \{b, c, d\} \nsubseteq \delta$-$pint(K) \cup \delta$-$pint(S) = \{c\} \cup \{d\} = \{c, d\}$.

Example 2. Consider the topological space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a, d\}, \{c\}, \{a, c, d\}\}$. If we take $M = \{d\}$ and $N = \{a\}$, then

$$\begin{aligned} \delta\text{-}pcl(M \cup N) &= \delta\text{-}pcl(\{a, d\}) \\ &= \{a, b, d\} \\ &\nsubseteq \delta\text{-}pcl(M) \cup \delta\text{-}pcl(N) \\ &= \{d\} \cup \{a\} \\ &= \{a, d\}. \end{aligned}$$

If we take $M = \{a, d\}$ and $N = \{b\}$, then

$$\begin{aligned} \delta\text{-}pcl(M) \cap \delta\text{-}pcl(N) &= \delta\text{-}pcl(\{a, d\}) \cap \delta\text{-}pcl(\{b\}) \\ &= \{a, b, d\} \cap \{b\} \\ &= \{b\} \\ &\nsubseteq \delta\text{-}pcl(M \cap N) \\ &= \delta\text{-}pcl(\{a, d\} \cap \{b\}) \\ &= \delta\text{-}pcl(\varnothing) \\ &= \varnothing. \end{aligned}$$

Definition 5. Let X be a topological space and $S \subset X$.
Then δ-$pcl(S) \backslash \delta$-$pint(S)$ is said to be the δ-prefrontier of S and is denoted by δ-$pfr(S)$.

Definition 6. Let X be a topological space and $S \subset X$.
Then $pcl(S) \backslash pint(S)$ is called the prefrontier of S and is denoted by $pfr(S)$ [6].

Remark 2. δ-$pfr(S) \subset pfr(S)$ for any subset S of a topological space X. The converse is not true in general.

Example 3. Let $\mathbb{R}$ be the set of real numbers endowith with the co-countable topology. Then $\delta PO(X)$ is the power set of X but no countable subsets of X are preopen . If we take $S = \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers, then $pfr(\mathbb{N}) = \mathbb{N} \nsubseteq \delta$-$pfr(\mathbb{N}) = \varnothing$.

Theorem 4. *Let X be a topological space and $S \subset X$. The following hold:*
(1) δ-$pcl(S) = \delta$-$pint(S) \cup \delta$-$pfr(S)$,
(2) δ-$pint(S) \cap \delta$-$pfr(S) = \varnothing$,
(3) δ-$pfr(S) = \delta$-$pcl(S) \cap \delta$-$pcl(X \backslash S)$.

Proof. (1) and (2) can be obtained by using definition of δ-prefrontier of S.

(3). By using definition of δ-prefrontier of S,

$$\begin{aligned}\delta\text{-}pfr(S) &= \delta\text{-}pcl(S)\backslash\delta\text{-}pint(S)\\ &= \delta\text{-}pcl(S)\cap(X\backslash\delta\text{-}pint(S))\\ &= \delta\text{-}pcl(S)\cap\delta\text{-}pcl(X\backslash S).\end{aligned}$$

□

REMARK 3. In view of the previous theorem, δ-$pfr(S)$ is a δ-preclosed set for any subset of X.

THEOREM 5. *Let X be a topological space and $S\subset X$. δ-$pfr(S)=\varnothing$ if and only if S is both a δ-preopen set and a δ-preclosed set.*

THEOREM 6. *The following hold for a subset S of a topological space X:*
(1) δ-$pfr(S)=\delta$-$pfr(X\backslash S)$,
(2) δ-$pfr(S)\subset X\backslash S$ if and only if $S\in\delta PO(X)$,
(3) δ-$pfr(S)\subset S$ if and only if $S\in\delta PC(X)$.

Proof. (1). It can be obtained easily.
(2). Let δ-$pfr(S)\subset X\backslash S$. Then δ-$pfr(S)\cap S=\varnothing$. Since δ-$pfr(S)=\delta$-$pcl(S)\cap(X\backslash\delta$-$pint(S))$, then δ-$pcl(S)\cap(X\backslash\delta$-$pint(S))\cap S=\varnothing$. We have $(X\backslash\delta$-$pint(S))\cap S=\varnothing$ and $S\subset\delta$-$pint(S)$. Hence, $S\in\delta PO(X)$.
Conversely, let $S\in\delta PO(X)$. Since δ-$pfr(S)=\delta$-$pcl(S)\backslash\delta$-$pint(S)$, then δ-$pfr(S)=\delta$-$pcl(S)\backslash S$. We have $S\cap\delta$-$pfr(S)=\varnothing$. Thus, δ-$pfr(S)\subset X\backslash S$.
(3). It is obvious. □

REMARK 4. Let M and N be subsets of a topological space X. $M\subset N$ does not imply that either δ-$pfr(M)\subset\delta$-$pfr(N)$ or δ-$pfr(N)\subset\delta$-$pfr(M)$.

EXAMPLE 4. Consider the topological space (X,τ) where $X=\{a,b,c,d\}$ and $\tau=\{X,\varnothing,\{a,d\},\{c\},\{a,c,d\}\}$. If we take $M=\{b\}$ and $N=\{a,b,c\}$, then $M\subset N$ and δ-$pfr(M)=\{b\}\nsubseteq\delta$-$pfr(N)=\varnothing$.
If we take $M=\{a\}$ and $N=\{a,c,d\}$, then $M\subset N$ and δ-$pfr(N)=\{b\}\nsubseteq\delta$-$pfr(M)=\varnothing$.

THEOREM 7. *Let S be a subset of a topological space X. The following hold:*
(1) δ-$pint(S)=S\backslash\delta$-$pfr(S)$,
(2) $X\backslash\delta$-$pfr(S)=\delta$-$pint(S)\cup\delta$-$pint(X\backslash S)$.

Proof. (1). Since δ-$pfr(S)=\delta$-$pcl(S)\backslash\delta$-$pint(S)$, then

$$\begin{aligned}\delta\text{-}pfr(S) &= \delta\text{-}pcl(S)\cap(X\backslash\delta\text{-}pint(S))\\ &= \delta\text{-}pcl(S)\cap\delta\text{-}pcl(X\backslash S).\end{aligned}$$

Thus,

$$\begin{aligned} S\backslash\delta\text{-}pfr(S) &= S\backslash[\delta\text{-}pcl(S)\cap\delta\text{-}pcl(X\backslash S)] \\ &= [S\backslash\delta\text{-}pcl(S)]\cup[S\backslash\delta\text{-}pcl(X\backslash S)] \\ &= \varnothing\cup[S\backslash\delta\text{-}pcl(X\backslash S)] \\ &= S\backslash\delta\text{-}pcl(X\backslash S) \\ &= S\cap[X\backslash\delta\text{-}pcl(X\backslash S)] \\ &= S\cap\delta\text{-}pint(S) \\ &= \delta\text{-}pint(S). \end{aligned}$$

(2). By the definition of δ-$pfr(S)$,

$$\begin{aligned} X\backslash\delta\text{-}pfr(S) &= X\backslash[\delta\text{-}pcl(S)\cap\delta\text{-}pcl(X\backslash S)] \\ &= [X\backslash\delta\text{-}pcl(S)]\cup[X\backslash\delta\text{-}pcl(X\backslash S)] \\ &= \delta\text{-}pint(X\backslash S)\cup\delta\text{-}pint(X\backslash X\backslash S) \\ &= \delta\text{-}pint(X\backslash S)\cup\delta\text{-}pint(S). \end{aligned}$$

□

THEOREM 8. *Let S be a subset of a topological space X. The following hold:*
(1) δ-$pfr(\delta$-$pint(S))\subset\delta$-$pfr(S)$,
(2) δ-$pfr(\delta$-$pcl(S))\subset\delta$-$pfr(S)$.

Proof. (1). By the definition of δ-$pfr(S)$,

$$\begin{aligned} \delta\text{-}pfr(\delta\text{-}pint(S)) &= \delta\text{-}pcl(\delta\text{-}pint(S))\backslash\delta\text{-}pint(\delta\text{-}pint(S)) \\ &= \delta\text{-}pcl(\delta\text{-}pint(S))\backslash\delta\text{-}pint(\delta). \end{aligned}$$

Since δ-$pint(\delta)\subset S$, then δ-$pcl(\delta$-$pint(S))\subset\delta$-$pcl(S)$. Thus,

$$\delta - pcl(\delta - pint(S))\backslash\delta - pint(S)\subset\delta - pcl(S)\backslash\delta - pint(S)$$

and

$$\delta - pfr(\delta - pint(S))\subset\delta - pcl(S)\backslash\delta - pint(S) = \delta - pfr(S).$$

(2). We have

$$\begin{aligned} \delta\text{-}pfr(\delta\text{-}pcl(S)) &= \delta\text{-}pcl(\delta\text{-}pcl(S))\backslash\delta\text{-}pint(\delta\text{-}pcl(S)) \\ &= \delta\text{-}pcl(S)\backslash\delta\text{-}pint(\delta\text{-}pcl(S)). \end{aligned}$$

Since $S\subset\delta$-$pcl(S)$, then δ-$pint(S)\subset\delta$-$pint(\delta$-$pcl(S))$. We obtain δ-$pcl(S)\backslash\delta$-$pint(\delta$-$pcl(S))\subset\delta$-$pcl(S)\backslash\delta$-$pint(S)$. Hence, δ-$pfr(\delta$-$pcl(S))\subset\delta$-$pfr(S)$. □

DEFINITION 7. Let X be a topological space and $x\in X$. x is called a prelimit point of $S\subset X$ if and only if $U\cap(S\backslash\{x\})\neq\varnothing$ for each preopen set U containing x. The set of all prelimit points of S is called the prederived set of S and is denoted by $pd(S)$ [6].

DEFINITION 8. Let X be a topological space, $S\subset X$ and $x\in X$. x is said to be a δ-prelimit point of S if and only if $G\cap(S\backslash\{x\})\neq\varnothing$ for each $G\in\delta PO(X,x)$.

REMARK 5. Every δ-prelimit point of a subset S of a topological space$X$ is a prelimit point of S.

Definition 9. Let X be a topological space. The set of all δ-prelimit points of $S \subset X$ is said to be the δ-prederived set of S and is denoted by δ-$pd(S)$.

Remark 6. Let $pd(S)$ be the prederived set of S. Then δ-$pd(S) \subset pd(S)$. Converse need not be true in general.

Example 5. Let $X = \{a, b, c\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Then b is a prelimit point of $S = \{b, c\}$, but it is not a δ-prelimit point of S. Thus, $pd(S) \nsubseteq \delta$-$pd(S)$.

Theorem 9. *Let X be a topological space and $S \subset X$. Then S is δ-preclosed if and only if δ-$pd(S) \subset S$.*

Proof. Let S be a δ-preclosed set and $x \notin S$. Since $x \in X \backslash S$ and $X \backslash S$ is a δ-preopen set, then $x \notin \delta$-$pd(S)$.

Conversely, let δ-$pd(S) \subset S$ and $x \in X \backslash S$. Since δ-$pd(S) \subset S$, then $x \notin S$ and so $x \notin \delta$-$pd(S)$. By the definition of δ-prederived set of S, there exists a δ-preopen set G containing x such that $G \cap (S \backslash \{x\}) = \varnothing$. Since $G \cap S = \varnothing$, then $G \subset X \backslash S$. Thus, $X \backslash S$ is a δ-preopen set and S is a δ-preclosed set. □

Theorem 10. *Let M and N be subsets of a topological space X. If $M \subset N$, then δ-$pd(M) \subset \delta$-$pd(N)$.*

Remark 7. The previous implication is not conversible.

Example 6. Consider the topological space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{b\}, \{d\}, \{b, d\}\}$. If we take $M = \{a, b, c\}$ and $N = \{a, c, d\}$, then δ-$pd(M) = \delta$-$pd(N) = \{a, c\}$, but $M \neq N$.

Theorem 11. *Let X be a topological space and $M \subset X$. $M \cup \delta$-$pd(M)$ is a δ-preclosed set.*

Proof. Let $x \notin M \cup \delta$-$pd(M)$. Then $x \notin M$ and $x \notin \delta$-$pd(M)$. Since $x \notin \delta$-$pd(M)$, then there exists a $G \in \delta PO(X, x)$ such that $G \cap (M \backslash \{x\}) = \varnothing$. Since $x \notin M$, then $G \cap M = \varnothing$ and $G \subset X \backslash M$. Moreover, $G \subset X \backslash \delta$-$pd(M)$. Hence, $G \subset X \backslash (M \cup \delta$-$pd(M))$. We obtain that $M \cup \delta$-$pd(M)$ is a δ-preclosed set. □

Theorem 12. *Let X be a topological space and $M \subset X$. Then δ-$pint(M) = M \backslash \delta$-$pd(X \backslash M)$.*

Proof. Let $x \notin M \backslash \delta$-$pd(X \backslash M)$. We have $x \in \delta$-$pd(X \backslash M)$. By definition of the δ-prederived set of $X \backslash M$, $G \cap [(X \backslash M) \backslash \{x\}] \neq \varnothing$ for each $G \in \delta PO(X, x)$. We obtain $G \cap (X \backslash M) \neq \varnothing$ and $G \nsubseteq M$ for each $G \in \delta PO(X, x)$. Thus, $x \notin \delta$-$pint(M)$.

Conversely, let $x \in M \backslash \delta$-$pd(X \backslash M)$. Then $x \in M$ and $x \notin \delta$-$pd(X \backslash M)$. Since $x \in M$ and $x \notin \delta$-$pd(X \backslash M)$, then there exists a $G \in \delta PO(X, x)$ such that $G \cap [(X \backslash M) \backslash \{x\}] = \varnothing$ and $G \cap (X \backslash M) = \varnothing$. Thus, $G \subset M$ and $x \in \delta$-$pint(M)$. □

THEOREM 13. *Let M and N be subsets of a topological space X. The following hold:*
(1) $\delta\text{-}pd(\varnothing) = \varnothing$,
(2) *If* $x \in \delta\text{-}pd(M)$, *then* $x \in \delta\text{-}pd(M \backslash \{x\})$,
(3) $\delta\text{-}pd(M) \cup \delta\text{-}pd(N) \subset \delta\text{-}pd(M \cup N)$,
(4) $\delta\text{-}pd(M \cap N) \subset \delta\text{-}pd(M) \cap \delta\text{-}pd(N)$.

Proof. (1). It is clear.
(2). Let $x \in \delta\text{-}pd(M)$.
By definition of δ-prederived set of M, $G \cap (M \backslash \{x\}) \neq \varnothing$ for each $G \in \delta PO(X, x)$. Thus, $x \in \delta\text{-}pd(M \backslash \{x\})$.
(3) and (4). It can be obtained easily. □

REMARK 8. Inclusions in (2) and (3) can not be replaced equalities.

EXAMPLE 7. Consider the topological space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a, d\}, \{c\}, \{a, c, d\}\}$. If we take $M = \{d\}$ and $N = \{a\}$, then

$$\begin{aligned} \delta\text{-}pd(M \cup N) &= \delta\text{-}pd(\{a, d\}) \\ &= \{b\} \\ &\not\subseteq \delta\text{-}pd(M) \cup \delta\text{-}pd(N) \\ &= \varnothing \cup \varnothing \\ &= \varnothing. \end{aligned}$$

EXAMPLE 8. Consider the topological space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{b\}, \{d\}, \{b, d\}\}$. If we take $M = \{a, b\}$ and $N = \{b, c\}$, then

$$\begin{aligned} \delta\text{-}pd(M) \cap \delta\text{-}pd(N) &= \delta\text{-}pd(\{c, d\}) \cap \delta\text{-}pd(\{b, c\}) \\ &= \{a, c\} \cap \{a, c\} \\ &= \{a, c\} \\ &\not\subseteq \delta\text{-}pd(M \cap N) \\ &= \delta\text{-}pd(\{c\}) \\ &= \varnothing. \end{aligned}$$

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