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A CLASS OF MODULES CHARACTERIZED BY WEAKLY ASSOCIATED PRIME IDEALS

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Abstract. Let R be a commutative ring with non-zero identity. For a non-empty set $\mathcal{P}_R$ of prime ideals of R, we study the class $\mathcal{C}_R$ of R-modules A with the property that each weakly associated prime ideal of A belongs to $\mathcal{P}_R$. We show that $\mathcal{C}_R$ is a torsion class for a hereditary torsion theory if and only if $\mathcal{C}_R = R\text{-}Mod$. Also, we prove that $\mathcal{C}_R$ is a torsionfree class for some hereditary torsion theory, provided R is Artinian.

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1. INTRODUCTION

Throughout the paper we denote by R a commutative ring with non-zero identity. All modules considered are in the category R-Mod of unital R-modules. Let A be a module. We denote by $E(A)$ the injective hull of A and by $\text{Ann}_R x$ the annihilator of an element $x \in A$ in R. Let p be a prime ideal of R. Then p is said to be an associated prime ideal of A if there exists $x \in A$ such that $p = \text{Ann}_R x$. Also, p is said to be a weakly associated prime ideal of A if there exists $x \in A$ such that p is a prime ideal which is minimal among the prime ideals containing $\text{Ann}_R x$. We shall denote by $\text{Ass}_R(A)$ (respectively $\widetilde{\text{Ass}}_R(A)$) the set of associated prime ideals of A (respectively weakly associated prime ideals of A). In general, we have the inclusion $\text{Ass}_R(A) \subseteq \widetilde{\text{Ass}}_R(A)$, where we have equality if R is Noetherian. But there exist non-Noetherian rings for which the equality holds and rings for which the above inclusion is strict [5]. In [4] and [5], Yassemi established some properties of the sets $\text{Ass}_R(A)$ and $\widetilde{\text{Ass}}_R(A)$ and studied when they are equal.

For a non-empty set $\mathcal{P}_R$ of prime ideals of R, denote by $\mathcal{C}_R$ the class of R-modules A for which $\widetilde{\text{Ass}}_R(A) \subseteq \mathcal{P}_R$. We show that $\mathcal{C}_R$ is closed under submodules, direct sums and extensions and $\mathcal{C}_R$ is a torsion class for a hereditary torsion theory if and only if $\mathcal{C}_R = R\text{-}Mod$. Also, we show that $\mathcal{C}_R$ is closed under direct limits and injective hulls, provided R is Noetherian, and under direct products, provided R is Artinian. Therefore, $\mathcal{C}_R$ is a torsionfree class for some hereditary torsion theory, provided R is Artinian.

2. THE CLASS $\mathcal{C}_R$

We shall recall first some results that will be used later in the paper.

THEOREM 2.1. [1, Chapter 4] , [5] *Let A be a module. Then the following hold:*

(i) $\mathrm{Ass}_R(A) \subseteq \widetilde{\mathrm{Ass}}_R(A)$.

(ii) *If either R or A is Noetherian, then* $\mathrm{Ass}_R(A) = \widetilde{\mathrm{Ass}}_R(A)$.

(iii) *If $0 \to A \to B \to C \to 0$ is a short exact sequence of modules, then*

$$\widetilde{\mathrm{Ass}}_R(A) \subseteq \widetilde{\mathrm{Ass}}_R(B) \subseteq \widetilde{\mathrm{Ass}}_R(A) \cup \widetilde{\mathrm{Ass}}_R(C)\,.$$

(iv) *If $A = \bigoplus_{i\in I} A_i$, then* $\widetilde{\mathrm{Ass}}_R(A) = \bigcup_{i\in I} \widetilde{\mathrm{Ass}}_R(A_i)$.

(v) *If S is a multiplicative closed system of R, then*

$$\widetilde{\mathrm{Ass}}_{S^{-1}R}(S^{-1}A) = \{pS^{-1}R \mid p \in \widetilde{\mathrm{Ass}}_R(A) \text{ with } p \cap S = \emptyset\}\,.$$

(vi) *If R is Noetherian and B is a finitely generated module, then*

$$\mathrm{Ass}_R(\mathrm{Hom}_R(B, A)) = \{p \in \mathrm{Ass}_R(A) \mid q \subseteq p \text{ for some } q \in \mathrm{Ass}_R(B)\}\,.$$

(vii) $\widetilde{\mathrm{Ass}}_R(A) \neq \emptyset$ *if and only if* $A \neq 0$.

PROPOSITION 2.2. [4] *Let A be a module such that each weakly associated prime ideal of A is finitely generated. Then* $\mathrm{Ass}_R(A) = \widetilde{\mathrm{Ass}}_R(A)$.

THEOREM 2.3. [4] *Let $\varphi : R \to S$ be a homomorphism of commutative rings and let A be an S-module. Then*

$$\{\varphi^{-1}(p) \mid p \in \mathrm{Ass}_S(A)\} \subseteq \mathrm{Ass}_R(A) \subseteq \widetilde{\mathrm{Ass}}_R(A) \subseteq \{\varphi^{-1}(p) \mid p \in \widetilde{\mathrm{Ass}}_S(A)\}\,.$$

In what follows, let $\mathcal{P}_R$ be a non-empty set of prime ideals of R and denote by $\mathcal{C}_R$ the class of R-modules with the property $\widetilde{\mathrm{Ass}}_R(A) \subseteq \mathcal{P}_R$. Obviously, $\mathcal{C}_R$ is closed under isomorphic copies.

THEOREM 2.4. (i) *The class $\mathcal{C}_R$ is closed under submodules, extensions and direct sums.*

(ii) *If $R \in \mathcal{C}_R$, then every projective module belongs to $\mathcal{C}_R$.*

(iii) *Let $R \in \mathcal{C}_R$. Then the class $\mathcal{C}_R$ is closed under homomorphic images if and only if $\mathcal{C}_R = R$-Mod.*

Proof. (i) Let $A \in \mathcal{C}_R$ and let D be a submodule of A. By Theorem 2.1, we have $\widetilde{\mathrm{Ass}}_R(D) \subseteq \widetilde{\mathrm{Ass}}_R(A) \subseteq \mathcal{P}_R$, hence $D \in \mathcal{C}_R$.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules with $A, C \in \mathcal{C}_R$. By Theorem 2.1, we have $\widetilde{\mathrm{Ass}}_R(B) \subseteq \widetilde{\mathrm{Ass}}_R(A) \cup \widetilde{\mathrm{Ass}}_R(C) \subseteq \mathcal{P}_R$, hence $B \in \mathcal{C}_R$.

Let $A = \bigoplus_{i\in I} A_i$. By Theorem 2.1, $\widetilde{\mathrm{Ass}}_R(A) = \bigcup_{i\in I} \widetilde{\mathrm{Ass}}_R(A_i) \subseteq \mathcal{P}_R$, hence $A \in \mathcal{C}_R$.

(ii) Using (i), every free module belongs to $\mathcal{C}_R$ as a direct sum of copies of R and then every projective module belongs to $\mathcal{C}_R$, as a submodule of a free module.

(iii) Suppose that $\mathcal{C}_R$ is closed under homomorphic images. By (ii), every module belongs to $\mathcal{C}_R$, as a homomorphic image of a free module. $\square$

Corollary 2.5. *Let $R \in \mathcal{C}_R$. Then the class $\mathcal{C}_R$ is a torsion class for some hereditary torsion theory if and only if $\mathcal{C}_R = R$-Mod.*

Proof. Note that a class of modules is a torsion class for a hereditary torsion theory if and only if it is closed under submodules, homomorphic images, direct sums and extensions [2, Proposition 1.7]. Now apply Theorem 2.4. □

Proposition 2.6. *Let R be an integral domain, $\mathcal{P}_R = \widetilde{\mathrm{Ass}}_R(R)$ and let A be a non-zero module. Then $A \in \mathcal{C}_R$ if and only if A is torsionfree.*

Proof. Clearly, 0 is a prime ideal of R and $\mathrm{Ann}_R r = 0$ for every $0 \neq r \in R$, hence $\mathcal{P}_R = \widetilde{\mathrm{Ass}}_R(R) = \{0\}$. Then $A \in \mathcal{C}_R$ if and only if $\widetilde{\mathrm{Ass}}_R(A) = \{0\}$ if and only if $\mathrm{Ann}_R x = 0$ for every $0 \neq x \in A$ if and only if A is torsionfree. □

Theorem 2.7. *Let A be the direct limit of a direct system $((A_i)_{i\in I}; (\varphi_i^j)_{i,j\in I})$ of modules with $A_i \in \mathcal{C}_R$ for every $i \in I$ and suppose that every weakly associated prime ideal of A is finitely generated. Then $A \in \mathcal{C}_R$.*

Proof. For every $i \in I$, denote by $\varphi_i : A_i \to A$ the canonical homomorphism. By Proposition 2.2, $\mathrm{Ass}_R(A) = \widetilde{\mathrm{Ass}}_R(A)$. We may suppose that $A \neq 0$. Let $p \in \mathrm{Ass}_R(A)$. Then there exists $0 \neq x \in A$ such that $p = \mathrm{Ann}_R x$. Hence there exists $x_i \in A_i$ for some $i \in I$ such that $\varphi_i(x_i) = x$.

We claim that $r \subseteq \mathrm{Ann}_R(x_j)$. Let $r \in p$. Then $\varphi_i(rx_i) = rx = 0$. Since p is finitely generated, there exists $j \in I$ with $j \geq i$ such that $\varphi_i^j(rx_i) = 0$. Denote $x_j = \varphi_i^j(x_i) \in A_j$. Since $rx_j = \varphi_i^j(rx_i) = 0$, we have $r \in \mathrm{Ann}_R(x_j)$. Hence $r \subseteq \mathrm{Ann}_R(x_j)$.

Now let $s \in \mathrm{Ann}_R(x_j)$. Since $x = \varphi_i(x_i) = \varphi_j(\varphi_i^j(x_i)) = \varphi_j(x_j)$, we have $sx = \varphi_j(sx_j) = \varphi_j(0) = 0$, hence $s \in p$. It follows that

$$p \subseteq \mathrm{Ann}_R(x_j) \in \mathrm{Ass}_R(A_j) \subseteq \widetilde{\mathrm{Ass}}_R(A_j) \subseteq \mathcal{P}_R\,.$$

Hence $\widetilde{\mathrm{Ass}}_R(A) \subseteq \mathcal{P}_R$, showing that $A \in \mathcal{C}_R$. □

Corollary 2.8. *Let R be Noetherian and let A be the direct limit of a direct system $((A_i)_{i\in I}; (\varphi_i^j)_{i,j\in I})$ of modules with $A_i \in \mathcal{C}_R$ for every $i \in I$. Then $A \in \mathcal{C}_R$.*

Corollary 2.9. *If R is Noetherian and $R \in \mathcal{C}_R$, then every flat module belongs to $\mathcal{C}_R$.*

Proof. Every flat module is a direct limit of a direct system of finitely generated free modules [3, Chapter 2, Theorem 1.2]. Now the conclusion follows by Theorem 2.4 and Corollary 2.8. □

Theorem 2.10. *Let A be a module and suppose that there exists a strictly increasing chain*

$$0 = A_0 \subset A_1 \subset \cdots \subset A_\alpha \subset A_{\alpha+1} \subset \cdots \subset A_\sigma = A$$

of submodules of A, for some ordinal σ, such that $A_{\alpha+1}/A_\alpha \in \mathcal{C}_R$ for every $0 \leq \alpha < \sigma$ and $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ for every limit ordinal $\beta \leq \sigma$. Then $A \in \mathcal{C}_R$.

Proof. For $\alpha = 0$, we have $A_1 \cong A_1/A_0 \in \mathcal{C}_R$. Let $1 \leq \alpha < \beta$ and assume that $A_\alpha \in \mathcal{C}_R$ for every $1 \leq \alpha < \beta$.

Suppose first that β is a succesor of an ordinal γ, that is, $\beta = \gamma + 1$, and consider the short exact sequence of modules $0 \to A_\gamma \to A_\beta \to A_\beta/A_\gamma \to 0$. We have $A_\gamma, A_\beta/A_\gamma \in \mathcal{C}_R$, hence $A_\beta \in \mathcal{C}_R$ by Theorem 2.4.

Now suppose that β is a limit ordinal. Then $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ and $A_\alpha \in \mathcal{C}_R$ for every limit ordinal $\alpha < \beta$. Let $p \in \widetilde{\mathrm{Ass}}_R(A_\beta)$. Then there exists $x \in A$ such that p is the minimal prime ideal that contains $\mathrm{Ann}_R x$. Also, there exists an ordinal $\delta < \beta$ such that $x \in A_\delta$. Hence $p \in \widetilde{\mathrm{Ass}}_R(A_\delta) \subseteq \mathcal{P}_R$. Therefore, $\widetilde{\mathrm{Ass}}_R(A_\beta) \subseteq \mathcal{P}_R$, that is, $A_\beta \subseteq \mathcal{C}_R$.

Consequently, by transfinite induction, $A = A_\sigma \in \mathcal{C}_R$. □

LEMMA 2.11. *Let R be such that every weakly associated prime ideal of R is finitely generated, let p be a prime ideal of R and let $\mathcal{P}_R = \mathrm{Ass}_R(R)$. Then $R/p \in \mathcal{C}_R$ if and only if there exists an ideal I of R such that $I \cong R/p$.*

Proof. By Proposition 2.2, $\mathrm{Ass}_R(R) = \widetilde{\mathrm{Ass}}_R(R)$.

Suppose first that $R/p \in \mathcal{C}_R$. Since $\widetilde{\mathrm{Ass}}_R(R/p) = \{p\}$, we have $p \in \widetilde{\mathrm{Ass}}_R(R)$. Hence there exists $0 \neq r \in R$ such that $\mathrm{Ann}_R r = p$. Then $R/p \cong I$, where $I = rR$.

Conversely, suppose that there exists an ideal I of R such that $I \cong R/p$. Since $\mathrm{Ann}_R x = p$ for every $x \in I$, it follows that $p \in \mathrm{Ass}_R(R)$. Therefore, $R/p \in \mathcal{C}_R$. □

EXAMPLE 2.12. Let R be semisimple such that $R \in \mathcal{C}_R$. Since R is Artinian, R is a finite direct sum of simple ideals and every prime ideal is maximal. By Theorem 2.1, $\mathrm{Ass}_R(R) = \widetilde{\mathrm{Ass}}_R(R)$ and it is the (finite) set of all maximal ideals of R, whence $\mathcal{P}_R = \mathrm{Ass}_R(R)$. Then, by Lemma 2.11, every simple module belongs to $\mathcal{C}_R$. Finally, by Theorem 2.4, it follows that every module belongs to $\mathcal{C}_R$.

THEOREM 2.13. *Let R be Artinian. Then the class $\mathcal{C}_R$ is closed under direct products.*

Proof. Let $A = \prod_{i\in I} A_i$, where each $A_i \in \mathcal{C}_R$. By Theorem 2.1, we have $\mathrm{Ass}_R(B) = \widetilde{\mathrm{Ass}}_R(B)$ for every module B. By hypothesis, every prime ideal of R is maximal and R has a finite number of maximal ideals. Without loss of generality, we may suppose that $A_i \neq 0$ for every $i \in I$. Let $p \in \widetilde{\mathrm{Ass}}_R(A)$. Then there exists $0 \neq x = (x_i)_{i\in I} \in A$, where $x_i \in A_i$ for each $i \in I$, such that $p = \mathrm{Ann}_R x$. We have $p = \bigcap_{i\in I} \mathrm{Ann}_R x_i$. Denote

$$J = \{i \in I \mid 0 \neq x_i \in A_i \text{ with } x = (x_i)_{i\in I}\}.$$

Clearly, $J \neq \emptyset$. It follows that $p = \bigcap_{i\in J} \mathrm{Ann}_R x_i$, hence $p \subseteq \mathrm{Ann}_R x_i$ for every $i \in J$. Since p is a maximal ideal of R and $\mathrm{Ann}_R x_i \neq R$, we have $p = \mathrm{Ann}_R x_i$ for every $i \in J$. Then $p \in \widetilde{\mathrm{Ass}}_R(A_i)$ for every $i \in J$, hence $p \in \mathcal{P}_R$. It follows that $\widetilde{\mathrm{Ass}}_R(A) \subseteq \mathcal{P}_R$, that is, $A \in \mathcal{C}_R$. □

Theorem 2.14. *Let R be Noetherian. Then the class $\mathcal{C}_R$ is closed under injective hulls.*

Proof. Let A be a module. If $A = 0$, then $E(A) = 0$ and $\widetilde{\mathrm{Ass}}_R(E(A)) = \emptyset$. Thus $E(A) \in \mathcal{C}_R$.

Suppose that $A \neq 0$. Let $p \in \widetilde{\mathrm{Ass}}_R(E(A))$. Hence there exists $0 \neq x \in E(A)$ such that $p = \mathrm{Ann}_R x$ and there exists $s \in R$ such that $0 \neq sx \in A$. If $r \in p$, then $rsx = 0$, hence $r \in \mathrm{Ann}_R(sx)$. Hence $p \subseteq \mathrm{Ann}_R(sx)$.

We claim that $p = \mathrm{Ann}_R(sx)$. Suppose that $p \neq \mathrm{Ann}_R(sx)$. Then there exists $t \in \mathrm{Ann}_R(sx)$ and $t \notin p$. We have $tsx = 0$, hence $ts \in p$. Since p is prime, $s \in p$. Then $sx = 0$, which is a contradiction. Therefore, $p = \mathrm{Ann}_R(sx)$.

Now we have $p \in \widetilde{\mathrm{Ass}}_R(A) \subseteq \mathcal{P}_R$. Therefore, $\widetilde{\mathrm{Ass}}_R(E(A)) \subseteq \mathcal{P}_R$, that is, $E(A) \in \mathcal{C}_R$. □

Corollary 2.15. *Let R be Artinian. Then the class $\mathcal{C}_R$ is a torsionfree class for some hereditary torsion theory.*

Proof. By Theorems 2.1, 2.13 and 2.14, the class $\mathcal{C}_R$ is closed under submodules, direct products, injective hulls and isomorphic copies, hence it is a torsionfree class for some hereditary torsion theory [2, Proposition 1.10]. □

Proposition 2.16. *Let R be Noetherian, let B be a finitely generated module and let $A \in \mathcal{C}_R$. Then* $\mathrm{Hom}(B, A) \in \mathcal{C}_R$.

Proof. By Theorem 2.1 (ii) and (vi), because $\widetilde{\mathrm{Ass}}_R(A) \subseteq \mathcal{P}_R$. □

Proposition 2.17. *Let S be a multiplicative closed system of R, let $\mathcal{P}_R = \widetilde{\mathrm{Ass}}_R(R)$, $\mathcal{P}_{S^{-1}R} = \widetilde{\mathrm{Ass}}_{S^{-1}R}(S^{-1}R)$ and let $A \in \mathcal{C}_R$. Then $S^{-1}A \in \mathcal{C}_{S^{-1}R}$.*

Proof. We have $\widetilde{\mathrm{Ass}}_R(A) \subseteq \mathcal{P}_R = \widetilde{\mathrm{Ass}}_R(R)$. By Theorem 2.1 (v), it follows that

$$\widetilde{\mathrm{Ass}}_{S^{-1}R}(S^{-1}R) = \{pS^{-1}R \mid p \in \widetilde{\mathrm{Ass}}_R(R) \text{ with } p \cap S = \emptyset\}\,.$$

Hence $\widetilde{\mathrm{Ass}}_{S^{-1}R}(S^{-1}A) \subseteq \widetilde{\mathrm{Ass}}_{S^{-1}R}(S^{-1}R) = \mathcal{P}_{S^{-1}R}$. Therefore, $S^{-1}A \in \mathcal{C}_{S^{-1}R}$. □

Proposition 2.18. *Let S be a commutative Noetherian ring and let $\varphi : R \to S$ be a homomorphism. Also, let $\mathcal{P}_S = \mathrm{Ass}_S(S)$, let A be an S-module such that $A \in \mathcal{C}_S$ and let $S \in \mathcal{C}_R$. Then $A \in \mathcal{C}_R$.*

Proof. By Theorem 2.1, $\mathrm{Ass}_S(A) = \widetilde{\mathrm{Ass}}_S(A)$. By Theorem 2.3,

$$\mathrm{Ass}_R(A) = \widetilde{\mathrm{Ass}}_R(A) = \{\varphi^{-1}(p) \mid p \in \mathrm{Ass}_S(A)\}\,.$$

We also have

$$\mathrm{Ass}_R(S) = \widetilde{\mathrm{Ass}}_R(S) = \{\varphi^{-1}(p) \mid p \in \mathrm{Ass}_S(S)\}\,.$$

But $\mathrm{Ass}_S(A) \subseteq \mathcal{P}_S = \mathrm{Ass}_S(S)$, because $A \in \mathcal{C}_S$. Hence $\mathrm{Ass}_R(A) \subseteq \mathrm{Ass}_R(S) \subseteq \mathcal{P}_R$, because $S \in \mathcal{C}_R$. It follows that $A \in \mathcal{C}_R$. □

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