MATHEMATICA, Tome 47 (70), N° 2, 2005, pp. 125–136

(CO)TRIPLES AND HOCHSCHILD (CO)HOMOLOGY OF SUPERALGEBRAS

PAUL A. BLAGA

Abstract. We show that the Hochschild homology and cohomology of a given (not necessarily commutative) $\mathbb{Z}_2$-graded algebra (superalgebra) R over a graded-commutative $\mathbb{Z}_2$-graded ring, with coefficients in an R-bimodule can be defined as the homology (cohomology) of a suitable chosen cotriple (triple), with coefficients in a functor.

MSC 2000. 17A70, 16E40, 18C15

Key words. Superalgebras, Hochschild homology, triples.

1. INTRODUCTION

The Hochschild homology of an algebra was introduced by Gerhard Hochschild ([6]) in 1945 and it soon became a standard tool for algebraist, for the investigation of the structure of an associative algebra and, in particular, of a ring. In the early 1980s, Alain Connes introduced his noncommutative spaces, whose studies amounts, actually, to the investigation of some noncommutative algebras and suggested that Hochschild (co)homology could play an important role in the process. It became clear, also, that Hochschild homology has closed connections to K-theory and, as such, today is an important tool in many branches of mathematics, beside the algebra itself. The Hochschild homology of superalgebras ($\mathbb{Z}_2$-graded algebras) came into the picture in the second part of the eighties (see, for instance, [8] or [9]). The initial definition was similar to the one for ungraded algebras (see [10]), being just an adaptation of the definition given by MacLane in the cited reference for arbitrary graded algebras. Kassel ([8]) shown that in the case when the ground ring is field (with the trivial grading), then the Hochschild homology of the superalgebra R is an (absolute) Tor functor over the enveloping superalgebra $R^e \equiv R \otimes R^{\mathrm{op}}$, where R^{op} is just R with the opposite multiplication (here *opposite* should be taken in the graded sense, see bellow.) In a previous paper ([3]) we proved that this is true in the general case, provided we substitute the absolute Tor functor with a relative one. In a similar manner, the Hochschild cohomology turns out to be a relative Ext functor over the enveloping algebra and, again, in good cases (for instance when the ground ring is a field), the attribute "relative" can be replaced by "absolute". We mention, however, that in some books (for instance in the classical book of Cartan and Eilenberg, [4]) the Hochschild homology is *defined* as an absolute Tor functor, therefore, for arbitrary ground rings, the notion introduced by Cartand and Eilenberg does not coincide with the (quasi) generally accepted notion.

In the late sixtieth, in a seminal paper ([1]) Barr and Beck showed that many homology theories can be constructed by using a functorial machinery called a *triple*. It is the aim of this paper to show that this is, also, the case of Hochschild homology of superalgebras.

2. SUPERALGEBRAS AND THEIR HOCHSCHILD (CO)HOMOLOGY

2.1. Superalgebras. We shall expose here only the very basic notions of superalgebra that will be used in the sequel. For details, one can use the books [2], [12].

Definition 1. A $\mathbb{Z}_2$-graded ring is a ring $(R, +, \cdot)$ which is $\mathbb{Z}_2$-graded as an Abelian group, i.e. it has two subgroups R_0 and R_1 such that

$$R = R_0 \oplus R_1$$

and, moreover, for any $\alpha, \beta \in \mathbb{Z}_2$, we have

$$R_\alpha R_\beta \subset R_{\alpha+\beta}.$$

The elements of R_i are called *homogeneous elements of R, of degree i*, for $i = 1, 2$. More specifically, the elements of R_0 are called *even elements*, while the elements of R_1 are called *odd elements* of R. If $x \in R_i$, we shall say that x is an element of *degree* or *parity* i and we shall write $|x| = i$.

Hereafter, all the graduation will be $\mathbb{Z}_2$-graduation, therefore we shall drop the prefix and we shall simple use the term “graded object”.

A graded ring R is called *graded-commutative, supercommutative* or, there is no danger of confusion, just *commutative*, if for any homogeneous elements $x, y \in R$, we have

$$x \cdot y = (-1)^{|x|\cdot|y|} y \cdot x.$$

Let R be a graded ring and M a left (right) R-module.

Definition 2. M is called a *graded* left (right) R-module if

(i) M is graded as an Abelian group, i.e. it has two subgroups M_0 and M_1 such that

$$M = M_0 \oplus M_1$$

and

(ii) the graduations of R and M are compatible, in the sense that, for any $\alpha, \beta \in \mathbb{Z}_2$, we have

$$R_\alpha M_\beta \subseteq M_{\alpha+\beta} \quad (\text{respectively} M_\alpha R_\beta \subseteq M_{\alpha+\beta}).$$

If, in particular, R is a commutative graded ring, then any graded left R-module M has a natural structure of graded right R-module, if we let

$$xr = (-1)^{|x|\cdot|r|} rx,$$

for any $r \in R$ and $x \in M$. Obviously, the converse is also true, therefore, if R is commutative, it makes sense to speak, simply, about *graded R-modules*, dropping the attributes “left” and “right”.

A map between two graded R-modules M and N is linear if it is linear in the ordinary, ungraded sense. However, we can introduce a grading on the R-module $\mathrm{Hom}_R(M,N)$, if we put, for $\alpha = 0, 1$,

$$\mathrm{Hom}_R(M,N)_\alpha = \{f \in \mathrm{Hom}_R(M,N), |f| = \alpha\},$$

where $|f| = \alpha$ if $f(M_\beta) \subset N_{\alpha+\beta}$, for any $\beta \in \mathbb{Z}_2$. Usually, a linear map between two graded R-module is called a *morphism* of graded modules if it has degree zero. It is easy to check that graded left (right) R-modules over a graded ring, with the corresponding morphisms (linear maps of degree zero) is a category.

A *graded $R - S$-bimodule*, where R and S are two graded rings is just an $R - S$-bimodule which is graded both as a left R-module and as a right S-module.

If M is a graded right R-module and N – a graded left R-module, their tensor product in the graded category is their ordinary, ungraded tensor product, with the grading given by

$$\begin{aligned}(M \otimes N)_0 &:= (M_0 \otimes N_0) \oplus (M_1 \otimes N_1),\\ (M \otimes N)_1 &:= (M_0 \otimes N_1) \oplus (M_1 \otimes N_0).\end{aligned}$$

The tensor product of linear maps between graded modules is, again, defined as in the classical case, up to a sign. Thus, let $f : M \to M'$ and $g : N \to N'$ be two homogeneous linear maps between graded right (respectively left) R-modules. Their tensor product is, by definition, the map

$$f \otimes g : M \otimes M' \to N \otimes N',$$

given, for each homogeneous elements $m \in M, n \in M$, by

$$(f \otimes g)(m \otimes n) := (-1)^{|g||m|} f(m) \otimes g(n).$$

We have now all the necessary ingredients to define the central objects of the paper, the *superalgebras*.

Definition 3. Let R be a graded-commutative $\mathbb{Z}_2$-graded ring. A $\mathbb{Z}_2$-graded R-algebra (or a *superalgebra* over R) is, by definition, a graded R-module A endowed with two linear morphisms of degree zero $\pi : A \otimes A \to A$ (the *multiplication*) and $I : R \to A$ (the unity) such that the following two diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\pi\otimes 1"] \arrow[d, "1\otimes\pi"'] & A \otimes A \arrow[d, "\pi"] \\ A \otimes A \arrow[r, "\pi"] & A \end{tikzcd} \qquad \begin{tikzcd} R \otimes A \arrow[r, equal] \arrow[d, "I\otimes 1"] & A \arrow[r, equal] \arrow[d, equal] & A \otimes R \arrow[d, "1\otimes I"] \\ A \otimes A \arrow[r, "\pi"] & A & A \otimes A \arrow[l, "\pi"'] \end{tikzcd}$$

If A and B are superalgebras over the same graded ring R, then they are, both, bimodules over R, therefore it makes sense to speak about their tensor product, as graded modules. Moreover, we can endow this tensor product with a structure of superalgebra over R.

Definition 4. Let A and B be two R-superalgebras. The *tensor product* of the two algebras is the tensor product of A and B, as graded R-modules, with the multiplication given by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb',$$

for any $a, a' \in A, b, b' \in B$, the unit being $1_A \otimes 1_B$.

It is not difficult to check that, indeed, $A \otimes B$ with this multiplication becomes a superalgebra.

Clearly, any superalgebra A over a graded ring R is, in particular, a graded ring itself, therefore it makes sense to speak about graded A-modules, which, restricting the scalars, are, in particular, also graded R-modules. We mention that for graded A-modules (over an R-superalgebra A) we have to kinds of tensor products, over A and over R. As a convention, all the tensor products without any subscript will be taken over the ground ring of the superalgebra.

2.2. The Hochschild homology and cohomology of a superalgebra. In the sequel we shall denote by k a fixed graded ring and by R a fixed superalgebra over k. We also choose an R-R-bimodule M. To define the Hochschild homology of the superalgebra R with values in the bimodule M, we proceed as in the ungraded case (see, for instance, [13]) and we consider a simplcial k-module $M \otimes R^*$, with $[n] \to M \otimes R^{\otimes n}$ and with the convention that $M \otimes R^{\otimes 0} \equiv M$.

For a fixed $n \in \mathbb{N}$, we define the face and degeneracy maps through

$$d_n^i(m \otimes r_1 \otimes \cdots \otimes r_n) = \begin{cases} mr_1 \otimes r_2 \otimes \cdots \otimes r_n & \text{if } i = 0 \\ m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n & \text{if } 0 < i < n \\ \epsilon_n \cdot r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1} & \text{if } i = n \end{cases}, \tag{1}$$

where $\epsilon_n = (-1)^{|r_n|(|m|+|r_1|+\cdots+|r_{n-1}|)}$, respectively

$$\sigma_n^i(m \otimes r_1 \otimes r_n) = m \otimes \ldots r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n. \tag{2}$$

It is easily checked that we defined, indeed, a simplicial module, as the simplicial identities are fulfilled. Therefore, we can proceed and define the Hochschild homology of R with coefficients in the R-module M as being the homology associated to the simplicial module (1) (2), in other words – the homology of the complex

$$0 \leftarrow M \xleftarrow{d_1} M \otimes R \xleftarrow{d_2} M \otimes R \otimes R \xleftarrow{d_3} \cdots, \tag{3}$$

where

$$d_n = \sum_{i=0}^{n} (-1)^i d_n^i. \tag{4}$$

Dually, to define the Hochschild cohomology, we can define first a cosimplicial k-module, by letting $[n] \to \mathrm{Hom}_k(R^{\otimes} n, M)$, with the convention that

$\mathrm{Hom}_k(R^{\otimes}n, M) \equiv M$ and defining the face and degeneracy mappings by

$$(5) \qquad (d_i^n f)(r_0, \ldots, r_n) = \begin{cases} r_0 f(r_1, \ldots, r_n) & \text{if } i = 0 \\ f(r_0, \ldots, r_{i-1}r_i, \ldots, r_n) & \text{if } 0 < i < n \,, \\ f(r_0, \ldots, r_{n-1}) r_n & \text{if } i = n \end{cases}$$

respectively

$$(6) \qquad \sigma_i^n(r_1, \ldots, r_{n-1}) = f(r_1, \ldots, r_i, 1, r_{i+1}, \ldots, r_{n-1}).$$

Then the Hochschild cohomology of R with coefficients in M is, by definition, the cohomology of the complex

$$(7) \qquad 0 \to M \xrightarrow{d^1} \mathrm{Hom}_k(R, M) \xrightarrow{d^2} \mathrm{Hom}_k(R \otimes R, M) \xrightarrow{d^3} \cdots \,,$$

where

$$(8) \qquad d^n = \sum_{i=0}^{n} (-1)^i d_i^n.$$

3. TRIPLES, COTRIPLES AND THE ASSOCIATED (CO)SIMPLICIAL OBJECTS

We recall (see, for instance, [13] or [11]) that if $\mathcal{A}$ is a category, then a *triple* in the category is a system (T, η, μ), where $T : \mathcal{A} \to \mathcal{A}$ is a functor, while $\eta : I_{\mathcal{A}} \to T$ and $\mu : T \to T^2$ are functorial morphisms such that the following diagrams commute:

$$(9) \qquad \begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

We mention that, in fact, the commutativity of the first diagram from (9) means that, for any $A \in \mathcal{A}$, we have

$$(10) \qquad \mu_A \circ T(\mu_A) = \mu_A \circ \mu_{T(A)},$$

while the meaning of the second one is that, again, for any $A \in \mathcal{A}$,

$$(11) \qquad \mu_A \circ \eta_{T(A)} = \mu_A \circ T(\eta_A) = 1_{T(A)}$$

Dually, a cotriple in a category $\mathcal{A}$ is a system (L, ε, δ), where $L : \mathcal{A} \to \mathcal{A}$ is a functor, while $\varepsilon : L \to I_{\mathcal{A}}$ and $\delta : L \to L^2$ are functorial morphisms such that the following diagrams commute:

$$(12) \qquad \begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L^2 \arrow[d, "L\delta"] \\ L^2 \arrow[r, "\delta L"] & L^3 \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl, equal] \arrow[d, "\delta"] \arrow[dr, equal] & \\ L & L^2 \arrow[l, "\varepsilon L"'] \arrow[r, "L\varepsilon"] & L \end{tikzcd}$$

As in the case of triples, after fixing an object $A \in \mathcal{A}$, the first diagram of (12) reads

$$L(\delta_A) \circ \delta_A = \delta_{L(A)} \circ \delta_A, \tag{13}$$

while the second one becomes

$$\varepsilon_{L(A)} \circ \delta_A = L(\varepsilon_A) \circ \delta_A = 1_{L(A)}. \tag{14}$$

The triples are also called *monad* and cotriples – *comonad* or *standard constructions*. Clearly, a cotriple is nothing but a triple in the opposite category of $\mathcal{A}$, $\mathcal{A}^{\mathrm{op}}$.

If (L, ε, δ) is a cotriple in an Abelian category $\mathcal{A}$, then, for any $A \in \mathcal{A}$ it determines a simplicial object if we put $C_n(A) := L^{n+1}(A)$ and we define the faces and degeneracy maps by

$$\partial_n^i := L^i(\varepsilon_{L^{n-i}(A)}) : L^{n+1}(A) \to L^n(A), \quad i = 0, \dots n \tag{15}$$

and, respectively,

$$s_n^i := L^i(\delta_{L^{n-i}(A)}) : L^{n+1}(A) \to L^{n+2}(A), \quad i = 0, \dots n. \tag{16}$$

It can be checked using the axioms of the cotriple (see, for instance, [13]), that all the identities of a simplicial object are, indeed, fulfilled for this family of objects and maps.

4. ADJOINT FUNCTORS AND (CO)TRIPLES

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. We recall that a pair of functors

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

is called an *adjoint pair* (or, else, it is said that G is a *left adjoint* of F), if there exist two functorial morphisms $\varepsilon : FG \to I_{\mathcal{B}}$ and $\eta : I_{\mathcal{A}} \to GF$ such that

$$\begin{cases} G\varepsilon \circ \eta G = 1_G \\ \varepsilon F \circ G\eta = 1_F \end{cases} . \tag{17}$$

i.e., for any fixed $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$\begin{cases} G(\varepsilon_B) \circ \eta_{G(B)} = 1_{G(B)} \\ \varepsilon_{F(A)} \circ G(\eta_A) = 1_{F(A)} \end{cases} . \tag{18}$$

It is a known fact ([13], [11]), that any adjoint pair $(F, G, \eta, \varepsilon)$, gives rise to a cotriple (and, hence, to a simplicial object) in the category $\mathcal{B}$, namely $L = (FG, \varepsilon, F\eta G)$.

5. THE BASE-CHANGE FUNCTOR AND ITS ASSOCIATED COTRIPLE

Let, hereafter, k be a fixed, commutative, superring and R a k-superalgebra (not necessarily commutative). We denote by $\mathcal{A}$ the category of $\mathbb{Z}_2$-graded k-modules and by $\mathcal{B}$ – the category of $\mathbb{Z}_2$-graded R-modules. Let $F : \mathcal{A} \to \mathcal{B}$ be the base change functor. In other words, for each k-module M, F is given by

$$F(M) = R \otimes_k M, \tag{19}$$

and for any k-modules M, N and each morphism $f : M \to N$, we have

$$F(f) \equiv 1_R \otimes f : R \otimes_k M \to R \otimes_k N. \tag{20}$$

As one checks easily, F has a left adjoint, the forgetful (underlying) functor $U : \mathcal{B} \to \mathcal{A}$. The unit of the adjunction is given by the functorial morphism $\eta : I_{\mathcal{A}} \to UF$, such that, for any k-module M,

$$\eta_M : M \to U(F(M)) \equiv R \otimes_k M, \quad \eta_M(m) = 1 \otimes m, \quad \forall m \in M, \tag{21}$$

while the counit is $\varepsilon : FU \to I_{\mathcal{B}}$ such that, for any R-module N we have

$$\varepsilon_N : FU(N) \equiv R \otimes_k U(N) \equiv R \otimes_k N \to N, \quad \varepsilon(r \otimes n) = rn, \quad \forall r \in R, n \in N. \tag{22}$$

We shall check now directly that, indeed, $L = (FU, \varepsilon, F\eta U)$ is a cotriple in the category of R-modules. In other words, we shall check that the identities (13) and (14) are fulfilled.

First of all, we notice that, for any R-module N, we have

$$L(N) \equiv FU(N) = R \otimes_k U(N) = R \otimes_k N, \tag{23}$$

where, in the right hand side, N is thought of with its underlying structure of k-module only, forgetting the initial R-module structure, while if $f : M \to N$ is a morphism of R-modules, then

$$L(f) = 1_R \otimes f : R \otimes_k M \to R \otimes_k N, \tag{24}$$

where, in the tensor product, f is thought of as a morphism of k-modules, instead of R-modules.

Now, let M be an arbitrary R-module. Then, according to the definition, we have

$$\delta_M : L(M) \equiv R \otimes M \to L^2(M) \equiv R \otimes R \otimes M,$$
$$\delta_M = F(\eta_{U(M)}) = 1_R \otimes \eta_{U(M)}.$$

In other words,

$$\delta_M(r \otimes m) = (1_R \otimes \eta_{U(M)})(r \otimes m) = 1_R(r) \otimes \eta_{U(M)}(m) = r \otimes 1 \otimes m,$$

for any $r \in R, m \in M$.

We start by evaluating the left hand side of (13). We have

$$(L(\delta_M) \circ \delta_M)(r \otimes m) = L(\delta_M)(r \otimes 1 \otimes m) = (1_R \otimes \delta_M)(r \otimes 1 \otimes m) =$$
$$= r \otimes 1 \otimes 1 \otimes m.$$

On the other hand, for the right hand side of the same equality, we obtain

$$\left(\delta_{L(M)} \circ \delta_M\right)(r \otimes m) = \delta_{R\otimes M}(r \otimes 1 \otimes m) = r \otimes 1 \otimes 1 \otimes m,$$

therefore (13) holds true in our case.

As for (14), we have, first of all,

$$(\varepsilon_{L(M)} \circ \delta_M)(r \otimes m) = \varepsilon_{R\otimes M}(r \otimes 1 \otimes m) = (r \otimes 1)m = r \otimes m \equiv 1_{L(M)}(r \otimes m).$$

On the other hand,

$$(L(\varepsilon_M) \circ \delta_M)(r \otimes m) = (1_R \otimes \varepsilon_M)(r \otimes 1 \otimes m) = r \otimes \varepsilon_M(1 \otimes m) = r \otimes m = \\ = 1_{L(M)}(r \otimes m),$$

therefore (14) holds true, as well.

We shall compute now the faces and the degeneracy maps of the symplicial object associated to the cotriple we built. We have

$$\partial_i^n : L^{n+1}(M) \equiv R^{\otimes(n+1)} \otimes M \to L^n(M) \equiv R^{\otimes n} \otimes M,$$
$$\partial_i^n = L^i(\varepsilon_{L^{n-i}(M)}).$$

More specifically, we have, for any $r_1 \otimes r_2 \otimes \cdots \otimes r_{n+1} \otimes m \in R^{\otimes(n+1)} \otimes M$,

$$\partial_i^n(r_0 \otimes r_1 \otimes \cdots \otimes r_n \otimes m) = L^i(\varepsilon_{L^{n-i}(M)})(r_0 \otimes r_1 \otimes \cdots \otimes r_n \otimes m) = \\ = r_0 \otimes \cdots \otimes r_{i-1} \otimes \varepsilon_{R^{\otimes(n-i)}\otimes M}(r_i \otimes r_{i+1} \otimes \cdots \otimes r_n \otimes m) = \\ = r_0 \otimes r_1 \otimes \cdots \otimes r_{i-1} \otimes r_i r_{i+1} \otimes r_{i+2} \otimes \cdots \otimes r_n \otimes m,$$

if $i = 0, 1, \ldots, n-1$, while

$$\partial_i^n(r_0 \otimes r_1 \otimes \cdots \otimes r_n \otimes m) = r_0 \otimes r_1 \otimes \cdots \otimes r_{n-1} \otimes r_n m.$$

As for the degeneracy maps, we have

$$s_i^n : L^{n+1}(M) \equiv R^{\otimes(n+1)} \otimes M \to L^{n+2}(M) \equiv R^{\otimes(n+2)} \otimes M,$$

$$s_i^n(r_0 \otimes r_1 \otimes \cdots \otimes r_n \otimes m) = L^i(\delta_{L^{n-i}(M)})(r_0 \otimes r_1 \otimes \cdots \otimes r_n \otimes m) = \\ = r_0 \otimes r_1 \otimes \cdots \otimes r_{i-1} \otimes \delta_{R^{\otimes(n-i)}\otimes M}(r_i \otimes \cdots \otimes r_n \otimes m) = \\ = r_0 \otimes \ldots r_{i-1} \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n \otimes m,$$

again, if $i = 0, \ldots, n-1$, and

$$s_n^n(r_0 \otimes r_1 \otimes \cdots \otimes r_n \otimes m) = r_0 \otimes r_1 \otimes \cdots \otimes r_n \otimes 1 \otimes m.$$

6. HOCHSCHILD HOMOLOGY OF A SUPERALGEBRA AS A COTRIPLE HOMOLOGY

Let (L, ε, δ) be a cotriple in a category $\mathcal{A}$, $\mathcal{M}$ – an Abelian category and $E : \mathcal{A} \to \mathcal{M}$ – a given functor. We give the following definition (see [1]):

Definition 5. The *cotriple homology* of an object A from $\mathcal{A}$ with coefficients in E, with respect to the given cotriple is the homology of the chain complex

$$0 \leftarrow E(L(A)) \overset{\delta}{\leftarrow} E(L^2(A)) \overset{\delta}{\leftarrow} E(L^3(A)) \leftarrow \ldots \tag{25}$$

Theorem 1. *Let R be a k-superalgebra and M – an R-bimodule. Then the Hochschild homology of R with coefficients in M is the cotriple homology of R associated to the base-change functor, with values in the functor $E \equiv M \otimes_{R\otimes R^{op}} -$.*

Proof. Let us denote, as customarily, by R^e the enveloping superalgebra $R \otimes R^{\mathrm{op}}$ of the superalgebra R. Then the chain complex (25) becomes

$$0 \leftarrow M \otimes_{R^e} R \otimes R \xleftarrow{1_M \otimes_{R^e} \partial^1} M \otimes_{R^e} R \otimes R \otimes R \leftarrow \cdots \leftarrow M \otimes_{R^e} R^{\otimes(n+2)} \leftarrow \cdots,$$

hence the n-chains are given by

$$C_n(R, M) = M \otimes_{R^e} R^{\otimes(n+2)}, \quad n = 0, 1, \ldots,$$

while the differentials are $\delta_n : C_n(R, M) \to C_{n-1}(R, M), \quad n = 1, 2, \ldots,$

$$\delta_n = \sum_{i=0}^{n} (-1)^i 1_M \otimes_{R^e} \partial_n^i.$$

More explicitly, we have

$$\delta_n(m \otimes_{R^e} r_0 \otimes r_1 \otimes \ldots r_{n+1}) = \sum_{i=0}^{n} (-1)^i m \otimes_{R^e} r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_{n+1}.$$

We shall show know that there is an isomorphism between the cotriple complex $(C(R, M), \delta)$ and the Hochschild complex $(A(R, M), d)$. As we saw in a previous paper, for each n there is an isomorphism θ_n between $C_n(R, M)$ and $A(R, M)$ given by

$$\begin{aligned} &\theta_n(m \otimes_{R^e} r_0 \otimes r_1 \otimes \cdots \otimes r_{n+1}) = (-1)^{|r_{n+1}|(|m|+|r_0|+\cdots+|r_n|)} \cdot \\ &\cdot r_{n+1} m r_0 \otimes r_1 \otimes \cdots \otimes r_n. \end{aligned} \tag{26}$$

θ_n is, clearly, linear and preserves the parity, therefore it is, indeed, a morphism of $\mathbb{Z}_2$-graded modules (supermodules). Its inverse, as it can be seen immediately, is given by $\theta_n^{-1} : A_n(R, M) \to C_n(R, M)$,

$$\theta_n^{-1}(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes_{R^e} 1 \otimes r_1 \otimes \cdots \otimes r_n \otimes 1. \tag{27}$$

The fact that the family of maps $\theta_n, n = 0, 1, \ldots$ is a morphism of complexes between the cotriple complex and the Hochschild complex simple means that for each n the following diagram is commutative

$$\begin{tikzcd} M \otimes_{R^e} R^{\otimes(n+2)} \arrow[r, "\theta_n"] \arrow[d, "1_M \otimes_{R^e} \partial_n"'] & M \otimes R^{\otimes n} \arrow[d, "d_n"] \\ M \otimes_{R^e} R^{\otimes(n+1)} \arrow[r, "\theta_{n-1}"] & M \otimes R^{\otimes(n-1)} \end{tikzcd}$$

or, in other words, that

$$d_n = \theta_{n-1} \circ (1_M \otimes_{R^e} \partial_n) \circ \theta_n^{-1}. \tag{28}$$

Clearly, it is enough to check that this relation is true on components, i.e. for faces:

$$d_n^i = \theta_{n-1} \circ (1_M \otimes_{R^e} \partial_n^i) \circ \theta_n^{-1}, \tag{29}$$

for $i = 0, 1, \ldots, n$. We shall make this check separately for $i = 0$, $1 \leq i \leq n-1$ and $i = n$.

For $i = 0$, we have

$$\begin{aligned}
&(\theta_{n-1} \circ (1_M \otimes_{R^e} \partial_n^0) \circ \theta_n^{-1})(m \otimes r_1 \otimes \cdots \otimes r_n) = \\
&= (\theta_{n-1} \circ (1_M \otimes_{R^e} \partial_n^0))(m \otimes_{R^e} 1 \otimes r_1 \otimes \cdots \otimes r_n \otimes 1) = \\
&= \theta_{n-1}(m \otimes_{R^e} r_1 \otimes \cdots \otimes r_n \otimes 1) = \\
&= (-1)^{|1|(|m|+|r_1|+\cdots+|r_n|+|1|)} 1 \cdot m \cdot r_1 \otimes r_2 \otimes \cdots \otimes r_n = \\
&= mr_1 \otimes r_2 \otimes \cdots \otimes r_n = d_n^0(m \otimes r_1 \otimes \cdots \otimes r_n).
\end{aligned}$$

Let us assume now that $1 \leq i \leq n-1$. Then

$$\begin{aligned}
&(\theta_{n-1} \circ (1_M \otimes_{R^e} \partial_n^i) \circ \theta_n^{-1})(m \otimes r_1 \otimes \cdots \otimes r_n) = \\
&= (\theta_{n-1} \circ (1_M \otimes_{R^e} \partial_n^i))(m \otimes_{R^e} 1 \otimes r_1 \otimes \cdots \otimes r_n \otimes 1) = \\
&= \theta_{n-1}(m \otimes_{R^e} 1 \otimes r_1 \otimes \cdots \otimes r_{i-1} \otimes r_i r_{i+1} \otimes r_{i+2} \otimes \cdots \otimes r_n \otimes 1) = \\
&= (-1)^{|1|(|m|+|r_1|+\cdots+|r_n|+|1|)} 1 \cdot m \cdot 1 \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n = \\
&= m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n = d_n^i(m \otimes r_1 \otimes \cdots \otimes r_n).
\end{aligned}$$

Finally, let $i = n$. We have

$$\begin{aligned}
&(\theta_{n-1} \circ (1_M \otimes_{R^e} \partial_n^n) \circ \theta_n^{-1})(m \otimes r_1 \otimes \cdots \otimes r_n) = \\
&= (\theta_{n-1} \circ (1_M \otimes_{R^e} \partial_n^n))(m \otimes_{R^e} 1 \otimes r_1 \otimes \cdots \otimes r_n \otimes 1) = \\
&= \theta_{n-1}(m \otimes_{R^e} 1 \otimes r_1 \otimes \cdots \otimes r_n) = \\
&= (-1)^{|r_n|(|m|+|r_1|+\cdots+|r_{n-1}|)} r_n \cdot m \cdot 1 \otimes r_1 \otimes \cdots \otimes r_{n-1} = \\
&= (-1)^{|r_n|(|m|+|r_1|+\cdots+|r_{n-1}|)} r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1} = \\
&= d_n^n(m \otimes r_1 \otimes \cdots \otimes r_n).
\end{aligned}$$

In fact, the isomorphisms θ_n provide – even more – an isomorphism of simplicial objects, in the simplicial category, because it can be checked, in exactly the same manner, that they are compatible, also, with the degeneracy maps, in other words we have, for each n,

$$\sigma_n^i = \theta_{n+1} \circ s_n^i \theta_n^{-1}, i = 0, \ldots, n$$

i.e. the following diagrams are commutative, for each n and each i:

$$\begin{tikzcd}
M \otimes_{R^e} R^{\otimes(n+2)} \arrow[r, "\theta_n"] \arrow[d, "1_M \otimes_{R^e} s_n^i"'] & M \otimes R^{\otimes n} \arrow[d, "\sigma_n^i"] \\
M \otimes_{R^e} R^{\otimes(n+3)} \arrow[r, "\theta_{n+1}"] & M \otimes R^{\otimes(n+1)}
\end{tikzcd} .$$

Thus, as the simplicial object associated to the cotriple and the Hochschild simplicial objects are isomorphic, they have the same homology, which concludes the proof. □

7. HOCHSCHILD COHOMOLOGY AS THE COHOMOLOGY OF A COTRIPLE

The coefficient functor used in the definition of the homology of a triple with coefficients in a functor is a *covariant* functor. If, instead, we apply a *contravariant* functor to the simplicial object associated to a cotriple L, we obtain a *cosimplicial* object. More precisely, we can give the following definition:

DEFINITION 6. Let $\mathcal{A}$ a category endowed with a cotriple (L, ε, δ), $\mathcal{M}$ – an Abelian category and $E : \mathcal{A} \to \mathcal{M}$ – a contravariant functor. The cotriple *cohomology* of an object A from $\mathcal{A}$ with respect to the cotriple L, with coefficients in the functor E is the cohomology of the complex of cochains

$$0 \to E(L(A)) \xrightarrow{\delta} E(L^2(A)) \xrightarrow{\delta} \dots$$

Let us return to ours superalgebra R and let, again, M be an R-bimodule. Then we have the following theorem, whose proof is analogous to the previous one and it is based on the construction of an isomorphism between the complex of cochains associated to the triple cohomology and the standard complex of cochains of Hochschild cohomology.

THEOREM 2. *The Hochschild cohomology of the superalgebra R with coefficients in the bimodule M is the cotriple cohomology associated to the base-change cotriple, with coefficients in the contravariant functor*

$$E \equiv \mathrm{Hom}_{R^e}(-, M).$$

Acknowledgment. This paper was partially supported by the European Commission, through the Research Training Network "Geometric Analysis".

REFERENCES

[1] BARR, M. and BECK, J., *Homology and Standard Constructions*, in Seminar on Triples and Categorical Homology Theory, LNM**80**, Springer, 1969.
[2] BARTOCCI, C., BRUZZO, U. and HERNANDEZ-RUIPEREZ, D., *The Geometry of Supermanifolds*, Kluwer, 1992.
[3] BLAGA, P.A., *Relative derived functors and Hochschild (co)homology for superalgebras*, in Proceedings of "Bolyai 200" International Conference on Geometry and Topology, Cluj-Napoca, 1–5 October 2003, edited by D. Andrica, P. Blaga, Z. Kása and F. Szenkovits, Cluj University Press, Cluj-Napoca (2003), 47–60.
[4] CARTAN, H. and EILENBERG, S., *Homological Algebra*, Princeton University Press, 1956.
[5] CONNES, A., *Noncommutative differential geometry*, Publications IHES 1985.
[6] HOCHSCHILD, G., *On the cohomology groups of an associative algebra*, Ann. of Math., **46** (1945), pp. 58–67.
[7] HOCHSCHILD, G., *Relative homological algebra*, Trans. AMS, **82** (1956), pp. 246–269.
[8] KASSEL, C., *A Künneth formula for the cyclic homology of $\mathbb{Z}/2$-graded algebras*, Math. Ann., **257** (1986), 683–699.

[9] Kastler, D., *Cyclic Cohomology within the Differential Envelope*, Hermann, Paris, 1988.
[10] MacLane, S., *Homology*, Springer Verlag, 1963.
[11] MacLane, S., *Categories for the Working Mathematician*, Springer Verlag, 1971.
[12] Manin, Yu., *Complex Geometry and Gauge Theory*, Springer Verlag, 1988.
[13] Weibel, Ch., *An Introduction to Homological Algebra*, Cambridge University Press, 1994.

Received November 15, 2004

"Babeş-Bolyai" University
Faculty of Mathematics and Computer Science
1, Kogălniceanu Street,
400084 Cluj-Napoca, Romania
E-mail: pablaga@cs.ubbcluj.ro