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CONDITIONAL CAUCHY EQUATIONS OF RIGHT CYLINDER TYPE ON n-GROUPS

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Abstract. The subject of this paper is the extension of the results obtained for a conditional Cauchy equation on groups (equation that is called by J. Dhombres [3], Cauchy equation of right cylinder type) to the similar equations on n-groups.

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1. INTRODUCTION

We recall the results for the Cauchy functional equation of right cylinder type on groups, that will be further used.

DEFINITION 1.1. ([3]) If $(G,\circ)$ and $(H,*)$ are two groups and Y is a subset of G, then the functional equation:

$$\begin{cases} f : G \to H \\ f(x\circ y) = f(x) * f(y),\ x \in G,\ y \in Y \end{cases}$$

is called conditional Cauchy equation of right cylinder type.

THEOREM 1.1. ([3]) *If $(G,\circ)$ and $(H,*)$ are two groups, Y is a nonempty subset of G, $G_\circ$ is the subgroup generated by Y in G, then the functional equations*

$$\begin{cases} f : G \to H \\ f(x\circ y) = f(x) * f(y),\ x \in G,\ y \in Y \end{cases} \tag{1.1}$$

and

$$\begin{cases} f : G \to H \\ f(x\circ y) = f(x) * f(y),\ x \in G,\ y \in G_\circ \end{cases} \tag{1.2}$$

are equivalent.

THEOREM 1.2. ([3]) *The general solution of the equation* (1.1) *(of right cylinder type) is:*

$$f(x) = h(p(x)) * g(x \circ (s(p(x)))^{-1}),$$

where:

$p : G \to G/\rho$ is the canonical projection on the quotient set with respect to the equivalence relation $x\rho y \iff x\circ y^{-1} \in G_\circ$;

$h : G/\rho \to H$ is an arbitrary function such that $h(p(1)) = 1$;

$s : G/\rho \to G$ is a lifting relative to p;

$g : G_\circ \to H$ is a morphism such that $g(s(p(1))) = 1$.

2. MAIN RESULTS

Let (G,φ) and (H,ψ) be $(n+1)$-groups with the $(n+1)$-ary operations $\varphi : G^{n+1} \to G$ and $\psi : H^{n+1} \to H$.

DEFINITION 2.1. If Y is a nonempty subset of G, then the functional equation:

$$(2.1)\qquad \begin{cases} f : G \to H \\ f(\varphi(x, y_1, \dots, y_n)) = \psi(f(x), f(y_1), \dots, f(y_n)) \end{cases}$$
$$x \in G, \quad y_1, \dots, y_n \in Y$$

is called Cauchy functional equation of right cylinder type.

REMARK 2.1. The Cauchy functional equation of right cylinder type is a constant conditional equation or a Z-conditional Cauchy equation [6], in which the set Z is $Z = G \times Y^n$.

In the sequel we will consider the equation (2.1), we denote by $G_\circ$ the sub-$(n+1)$-group generated by the set Y in G, $\overline{Z} = G \times G_\circ^n$ and the constant conditional Cauchy equation:

$$(2.2)\qquad \begin{cases} f : G \to H \\ f(\varphi(x, z_1, \dots, z_n)) = \psi(f(x), f(z_1), \dots, f(z_n)) \end{cases}$$
$$x \in G, \quad z_1, \dots, z_n \in G_\circ.$$

THEOREM 2.1. *The functional equations* (2.1) *and* (2.2) *are equivalent.*

Proof. In the proof we will use the notation

$$\varphi(x, y_1, \dots, y_n) = (x, y_1, \dots, y_n)_\circ$$
$$\psi(u, v_1, \dots, v_n) = (u, v_1, \dots, v_n)_*$$

We consider the sets

$$Z_1 = G \times Y^{n-1} \times G_\circ,\ Z_2 = G \times Y^{n-2} \times G_\circ^2, \dots, Z_n = G \times G_\circ^n = \overline{Z}$$

and we will prove by induction that all these Z_i-Cauchy conditional equations are equivalent.

Taking into account the inductive construction of the sub $(n+1)$-group generated by the set Y, to prove that the Z Cauchy equation is equivalent to the Z_1-Cauchy equation it is sufficient to show that:

a) If $x \in G$, $y_1, \dots, y_{n-1} \in Y$, $z_1, \dots, z_{n+1} \in Y$ and $z = (z_1, \dots, z_{n+1})_\circ$ then:

$$f((x, y_1, \dots, y_{n-1}, z)_\circ) = (f(x), f(y_1), \dots, f(y_{n-1}), f(z))_*;$$

b) If $x \in G$, $y_1, \dots, y_n \in Y$ then:

$$f((x, y_1, \dots, y_{n-1}, \overline{y}_n)_\circ) = (f(x), f(y_1), \dots, f(y_n), f(\overline{y}_n))_*$$

where $\overline{z}$ is the skew element of z in G.

a) $f((x, y_1, \dots, y_{n-1}, z)_\circ) = f(((x, y_1, \dots, y_{n-1}, z_1)_\circ, z_2, \dots, z_{n+1})_\circ) =$

$$= (f((x, y_1, \dots, y_{n-1}, z)_\circ), f(x_2), \dots, f(x_{n+1}))_* =$$

$$= ((f(x), f(y_1), \dots, f(y_{n-1}), f(z_1))_*, f(z_2), \dots, f(z_{n+1}))_* =$$
$$= (f(x), f(y_1), \dots, f(z))_*.$$

b) For $y \in G$ we can write

$$f(y) = f((\overline{y}, y, \dots, y)_0) = (f(\overline{y}), f(y), \dots, f(y))_*$$

and

$$(\overline{f(y)}, f(y), \dots, f(y))_* = f(x) \;\Rightarrow\; f(\overline{y}) = \overline{f(y)}.$$

Then:

$$f((x, y_1, \dots, y_n)_\circ) = f((x, y_1, \dots, y_{n-1}, (\overline{y}_n, \dots, y_n)_\circ)_\circ) =$$
$$= f(((x, y_1, \dots, y_{n-1}, \overline{y}_n)_\circ, y_n, \dots, y_n)_\circ) =$$
$$= f((x, y_1, \dots, y_{n-1}, \overline{y}_n)_\circ, f(y_n), \dots, f(y_n))_*.$$

We can also write:

$$f((x, y_1, \dots, y_n)_\circ) = (f(x), f(y_1), \dots, f(y_n))_* =$$
$$= (f(x), f(y_1), \dots, f(y_{n-1}), (\overline{f(y_n)}, f(y_n), \dots, f(y_n))_*)_* =$$
$$= ((f(x), f(y_1), \dots, f(y_{n-1}), \overline{f(y_n)})_\circ, f(y_n), \dots, f(y_n))_*.$$

From the previous two relations it follows:

$$f((x, y_1, \dots, y_{n-1}, \overline{y}_n)_\circ) = (f(x), f(y_1), \dots, f(y_{n-1}), f(\overline{y}_n))_*.$$

We prove now that the Z_1-Cauchy equation is equivalent with Z_2-Cauchy equation.

For $x \in G$, $y_1, \dots, y_{n-2} \in Y$, $z_1, \dots, z_{n+1} \in Y$, $z = (z_1, \dots, z_{n+1})_\circ$ and $t \in G_0$ we have:

$$f((x, y_1, \dots, y_{n-2}, z, t)_\circ) =$$
$$= f(((x, y_1, \dots, y_{n-2}, z_1, z_2)_\circ, z_3, \dots, z_{n+1}, t)_\circ) =$$
$$= (f((x, y_1, \dots, y_{n-2}, z_1, z_2)_\circ), f(z_3), \dots, f(z_{n+1}), f(t))_* =$$
$$= (f(x), f(y_1), \dots, f(y_{n-2}), f(z_1), f(z_2), \dots, f(z_{n+1}), f(t))_{**} =$$
$$= (f(x), f(y_1), \dots, f(y_{n-2}), f(z), f(t))_*.$$

For $x \in G$, $y_1, \dots, y_{n-1} \in Y$, $t \in G_\circ$; because $G_\circ$ is a sub-$(n+1)$-group the equation $(y_{n-1}, \dots, y_{n-1}, y)_0 = t \;\Leftrightarrow\; (\underset{n}{y_{n-1}}, y)_0 = t$ has a solution $y = u \in G_\circ$ and then:

$$\begin{aligned} f((x, y_1, \dots, \overline{y}_{n-1}, t)_\circ) &= f((x, y_1, \dots, \overline{y}_{n-1}, \underset{n}{y_{n-1}}, u)_{\circ\circ}) \\ &= f((x, y_1, \dots, y_{n-1}, u)_\circ) \\ &= (f(x), f(y_1), \dots, f(y_{n-1}), f(u))_* \\ &= (f(x), f(y_1), \dots, \overline{f(y_{n-1})}, f(y_{n-1}), \dots, f(y_{n-1}), f(u))_{**} \\ &= (f(x), f(y_1), \dots, \overline{f(y_{n-1})}, f(t))_* \\ &= (f(x), f(y_1), \dots, f(\overline{y}_{n-1}), f(t))_*. \end{aligned}$$

In the same way one can prove that the Z_i-Cauchy equation is equivalent to Z_{i+1}-Cauchy equation, for $i = 2, \dots, n-1$. $\square$

Foreword to obtain the general solution of the equation (2.1), we define an equivalence relation on G and we point out some properties of this relation.

If $G_\circ \le G$ is a sub-$(n+1)$-group in G, we define the relation: $(G, G, (G_\circ))$ by:

$x(G_\circ)y \;\Leftrightarrow\;$ there exists $u \in G_0$ such that $(x, \underset{n-2}{y}, \overline{y}, u) \in G_\circ$.

The relation $(G_\circ)$ has the properties [7]:

a) If $(G, \circ) = Red_u(G, (\;)_\circ)$ is the binary Hosszu reduces group and we denote by y^u the inverse element of y in $(G, \circ)$, then:

$$x(G_\circ)y \;\Leftrightarrow\; x \circ y^u \in G_\circ;$$

b) $x(G_\circ)y \;\Leftrightarrow\;$ there exists $g_1, \dots, g_n \in G_\circ$ such that:

$$x = (g_1, \dots, g_n, y)_\circ;$$

c) $x(G_\circ)y \;\Leftrightarrow\; (x, \underset{n-2}{y}, \overline{y}, v) \in G_\circ$ for all $v \in G_\circ$.

Let $u \in G_\circ$, $(G, \circ) = Red_u(G, (\;)_\circ)$, $(H, *) = Red_{f(u)}(H, (\;)_*)$, α, β the automorphisms of Hosszu's reduces:

$$\alpha(x) = (u, x, \underset{n-2}{u}, \overline{u})_\circ, \quad \beta(x) = (f(u), y, f(\underset{n-2}{u}), \overline{f(u)})_*$$

Since $\alpha \in \mathrm{Aut}(G, \circ)$ and $G_\circ$ is a sub-$(n+1)$-group, α is also an automorphism of $(G_\circ, \circ)$.

THEOREM 2.2. *The function $f : G \to H$ is a solution of equation* (2.2) *if an only if f is a solution of equation* (1.2) *and $f|_{G_\circ} : G_\circ \to H$ is $(n+1)$-groups morphism.*

Proof. If

$$f((x, g_1, \dots, g_n)_\circ) = (f(x), f(g_1), \dots, f(g_n))_*$$

then $f|_{G_\circ} : G_\circ \to H$ is $(n+1)$-groups morphism, $f(\overline{u}) = \overline{f(u)}$, $u \in G_\circ$, $\overline{u} \in G_\circ$, and

$$f((x, u_{n-2}, \overline{u}, g)_\circ) = (f(x), f(u_{n-2}) f(\overline{u}), f(g))_*,$$

$$x \in G, \quad u \in G_\circ, \quad g \in G_\circ.$$

Conversely:

$$(x, g_1, \dots, g_n)_\circ = x \circ \alpha(g_1) \circ \dots \circ \alpha^n(g_n) \circ a,$$

$$a = (\underset{n+1}{u})_\circ, \quad f(a) = f((\underset{n+1}{u})_\circ) = (\underset{n+1}{f(u)})_* = b,$$

$$f((x, g_1, \dots, g_n)_\circ) = f(x \circ g) = f(x) * f(g)$$

$$(g = \alpha(g_1) \circ \dots \circ \alpha^n(g_n) \circ a \in G_\circ).$$

But $f \circ \alpha|_{G_\circ} = \beta \circ f|_{G_\circ}$ [5] and

$$\begin{aligned} f(g) &= \beta(f(g_1 \circ \cdots \circ \alpha^{n-1}(g_n) \circ a) \\ &= \beta(f(g_1) * \cdots * f(\alpha^{n-1}(g_n)) * b \\ &= \beta(f(g_1)) * \cdots * \beta^n(f(g_n)) * b. \end{aligned}$$

It follows that $f(x) * f(g) = (f(x), f(g_1), \ldots, f(g_n))_*$. □

Applying Theorem 1.2 we obtain:

THEOREM 2.3. *The general solution of functional Cauchy equation of right cylinder type* (2.1) *is:*

$$f(x) = g(x \circ (s(p(x)))^{-1}) * h(p(x)),$$

where the functions p, h, s, g *are the same as those of* Theorem 1.2 *and the relation* ρ *is* $\rho = (G_\circ)$.

The theorem can be rewritten without using the reduction to bigroups, it is sufficient to take into account the relation between reduces and extendings.

THEOREM 2.4. *The function* $f : G \to H$ *is a solution of equation* (2.1) *if and only if there exist* $u \in G_\circ$, $v \in H$; $(v = g(u))$ *such that:*

$$f(x) = (g((x, s(\underset{n-2}{p}(x)), \overline{s(p(x))}, u)_\circ), \underset{n-2}{v}, \overline{v}, h(p(x)))_*,$$

where the functions p, s, h, g *are those from* Theorem 2.3.

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