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A DIRECT WAY TO OBTAIN STRONG DUALITY RESULTS IN LINEAR SEMIDEFINITE AND LINEAR SEMI-INFINITE PROGRAMMING

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Abstract. In linear programming it is known that an appropriate non-homogeneous Farkas Lemma leads to a short proof of the strong duality results for a pair of primal and dual programs. By using a corresponding generalized Farkas lemma we give a similar proof of the strong duality results for semidefinite programs under constraint qualifications. The proof also provides optimality conditions. The same approach leads to corresponding results for linear semi-infinite programs. For completeness, the proofs for linear programs and the proofs of all auxiliary lemmata for the semidefinite case are included.

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1. STRONG DUALITY RESULTS IN LINEAR PROGRAMMING

Consider the pair of primal and dual linear programs,

$$\begin{aligned} \mathrm{P}: &\quad \max_{x\in\mathbb{R}^n} \ c^{\mathrm{T}}x \quad \text{s.t.} \quad Ax \le b \\ \mathrm{D}: &\quad \min_{y\in\mathbb{R}^m} \ b^{\mathrm{T}}y \quad \text{s.t.} \quad A^{\mathrm{T}}y = c\,, \quad y \ge 0\,, \end{aligned}$$

where A is an $(m{\times}n)$-matrix ($m \ge n$) and $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Let v_P denote the maximum value of the primal program P and v_D the minimum value of the dual problem D. The feasible sets of P and D are abbreviated by F_P and F_D. Most commonly a homogeneous Farkas Lemma is used to prove optimality conditions for P and D. We will use the following non-homogeneous version to prove in one step existence of solutions, strong duality results and optimality conditions.

LEMMA 1. *Let be given an $(m\times n)$-matrix B, an $(k\times n)$-matrix C and $b\in\mathbb{R}^m$, $c\in\mathbb{R}^k$. Then precisely one of the following alternatives is valid.*

(a) *There is a solution $x\in\mathbb{R}^n$ of $Bx\le b$, $Cx=c$.*

(b) *There exist vectors $\mu\in\mathbb{R}^m$, $\mu\ge 0$, $\lambda\in\mathbb{R}^k$ such that*

$$\begin{pmatrix} B^{\mathrm{T}} \\ b^{\mathrm{T}} \end{pmatrix}\mu + \begin{pmatrix} C^{\mathrm{T}} \\ c^{\mathrm{T}} \end{pmatrix}\lambda = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

This result is an easy corollary of a common version of Farkas Lemma (see [1] for a proof). We begin with the weak duality result.

Lemma 2. (Weak Duality) *Let be given $x \in F_P$, $y \in F_D$. Then*

$$b^{\mathrm{T}}y - c^{\mathrm{T}}x = y^{\mathrm{T}}(b - Ax) \geq 0. \tag{1}$$

If in (1) *we have $b^{\mathrm{T}}y - c^{\mathrm{T}}x = 0$, then x, y are solutions of P, D with $v_P = v_D$.*

A proof of the weak duality lemma can be found in [1].

We now present the strong duality result, the existence of solutions and optimality conditions.

Theorem 1. (Strong Duality) *The following hold:*

(a) *Suppose $F_P \neq \emptyset$. Then $F_D = \emptyset$ if and only if $v_P = \infty$.*
Suppose $F_D \neq \emptyset$. Then $F_P = \emptyset$ if and only if $v_D = -\infty$.

(b) *Suppose F_P, $F_D \neq \emptyset$. Then P and D have solutions x and y satisfying $c^{\mathrm{T}}x = b^{\mathrm{T}}y$, i.e. $v_P = v_D$. Moreover, the following optimality conditions hold*

$$x \in F_P \text{ solves } P \iff \text{there exists } y \in F_D \text{ such that } y^{\mathrm{T}}(b - Ax) = 0 \ ,$$

$$y \in F_D \text{ solves } D \iff \text{there exists } x \in F_P \text{ such that } y^{\mathrm{T}}(b - Ax) = 0 \ .$$

A proof of the strong duality theorem can be found in [1].

2. STRONG DUALITY RESULTS IN SEMIDEFINITE PROGRAMMING

In this section we will give a similar proof of the strong duality result and optimality conditions in semidefinite programming. Consider the pair of primal and dual linear semidefinite programs

$$\begin{aligned} \mathrm{P}: &\quad \max_{x \in \mathbb{R}^n} \ c \circ x \quad \text{s.t.} \quad A(x) := B - \sum_{i=1}^{n} x_i A_i \succeq 0 \\ \mathrm{D}: &\quad \min_{Y} \ B \circ Y \quad \text{s.t.} \quad A_i \circ Y = c_i \ , i = 1, \ldots, n \ , \quad Y \succeq 0 \ , \end{aligned}$$

where B, A_i are symmetric $(m \times m)$-matrices and $c \in \mathbb{R}^n$. We write $Y \succeq 0$ for a positive semidefinite, and $Y \succ 0$ for a positive definite matrix Y. By $B \circ Y$ we denote the inner product $B \circ Y = \sum_{ij} b_{ij} y_{ij}$ (coinciding with the trace of BY). For convenience of notation we also have replaced $c^{\mathrm{T}}x$ by $c \circ x$. Let again v_P, v_D be the maximum, minimum values of P, D, respectively and F_P, F_D the feasible sets. Points $x \in F_P$, $Y \in F_D$ are called strictly feasible if $A(x)$, Y are positive definite. We firstly present a generalized non-homogeneous Farkas Lemma (see Section 4 for a proof). For a given set S let cone (S) denote the convex cone, lin (S) the linear hull and clos (S) the closure of S.

Lemma 3. *Let be given $S_0 = \{(b_k, \beta_k) \mid b_k \in \mathbb{R}^q, \ \beta_k \in \mathbb{R}, \ k \in K\}$, K a possibly infinite set, and $S_1 = \{(c_j, \gamma_j), \ c_j \in \mathbb{R}^q, \ \gamma_j \in \mathbb{R}, \ j \in J\}$, J a finite set. Then precisely one of the following alternatives is valid with $S :=$ cone $(S_0) +$ lin (S_1).*

(a) *There is a solution* ξ *of* $\quad b_k^{\mathrm{T}}\xi \le \beta_k,\ k\in K, \quad c_j^{\mathrm{T}}\xi = \gamma_j,\ j\in J.$

(b) $\binom{0}{-1} \in \mathrm{clos}\,(S)$.

We need a result for semidefinite matrices. A proof is given in Section 4.

LEMMA 4. *Let be given* $A,\ B\succeq 0$. *Then* $A\circ B \ge 0$ *and* $A\circ B = 0$ *if and only if* $A\cdot B = 0$. *If moreover* $A\succ 0$ *then* $A\circ B = 0 \Leftrightarrow B = 0$.

We treat the semidefinite problem as a direct generalization of the linear case. This approach is based on the following observation. Let $\mathcal{V}^m$ denote the compact set $\mathcal{V}^m = \{V = vv^{\mathrm{T}} \mid v\in\mathbb{R}^m,\ ||v|| = 1\}$. Then, in view of $A\circ vv^{\mathrm{T}} = v^{\mathrm{T}}Av$, it follows

$$(2) \qquad A\succeq 0 \quad\Longleftrightarrow\quad A\circ V\ge 0 \quad \text{for all } V\in\mathcal{V}^m\ .$$

We now proceed as in the case of linear programs.

LEMMA 5. (Weak Duality) *Let be given* $x\in F_P,\ Y\in F_D$. *Then*

$$(3) \qquad B\circ Y - c\circ x = Y\circ A(x) \ge 0.$$

If in (1) *we have* $B\circ Y - c\circ x = 0$, *then* $x,\ Y$ *are solutions of* $P,\ D$ *with* $v_P = v_D$.

Proof. For feasible $x,\ Y$ we find $B\circ Y - c\circ x = B\circ Y - \sum_{i=1}^n x_iA_i\circ Y = Y\circ A(x)\ge 0$ (see Lemma 4) or $B\circ Y \ge c\circ x$. The equal sign implies that Y is minimal for D and x is maximal for P with the same value $B\circ Y = v_D = c\circ x = v_P$. □

We prove the strong duality results together with optimality conditions under usual constraint qualifications.

THEOREM 2. (Strong Duality) *The following hold.*

(a) *Suppose* P *is strictly feasible. Then* $F_D = \emptyset$ *if and only if* $v_P = \infty$. *Suppose* D *is strictly feasible. Then* $F_P = \emptyset$ *if and only if* $v_D = -\infty$.

(b) *Suppose* P *and* D *are strictly feasible. Then* P *and* D *have solutions* x *and* Y *satisfying* $c\circ x = B\circ Y$. *Moreover, the following optimality conditions hold*

$$x\in F_P \text{ solves } P \Longleftrightarrow \text{there exists } Y\in F_D \text{ such that } Y\cdot A(x) = 0$$
$$Y\in F_D \text{ solves } D \Longleftrightarrow \text{there exists } x\in F_P \text{ such that } Y\cdot A(x) = 0\ .$$

Proof. In P we can assume that $A_i,\ i = 1,\ldots,n$ are linearly independent. (a): Assuming $F_D \ne \emptyset$, then with $Y\in F_D$ we obtain from Lemma 5, $B\circ Y\ge v_P$, i.e. $v_P < \infty$. Suppose now that $F_D = \emptyset$, i.e. there is no solution Y of

$$A_i\circ Y = c_i, \quad i = 1,\ldots,n, \quad -Y\circ V\le 0, \text{ for all } V\in\mathcal{V}^m.$$

By Lemma 3, $\binom{0}{-1} \in \mathrm{clos}\,\big(\mathrm{cone}\,(-\mathcal{V}^m, 0)) + \mathrm{lin}\,\{(A_i, c_i), i = 1,\ldots,n\}\big)$, i.e. there exist $V_k^\nu\in\mathcal{V}^m,\ \mu_k^\nu\ge 0,\ k\in K_\nu,\ \lambda_i^\nu\in\mathbb{R}$ such that

$$\sum_{k\in K_\nu}\mu_k^\nu\begin{pmatrix}-V_k^\nu\\ 0\end{pmatrix} + \sum_{i=1}^n\lambda_i^\nu\begin{pmatrix}A_i\\ c_i\end{pmatrix}\to\begin{pmatrix}0\\ -1\end{pmatrix} \qquad \text{for } \nu\to\infty\ .$$

Putting $S^\nu = \sum_{k\in K_\nu} \mu_k^\nu V_k^\nu$ and $x^\nu = -\lambda^\nu$ this is equivalent to

$$(4)\quad -\sum_{i=1}^{n} x_i^\nu A_i + E^\nu = S^\nu \succeq 0,\ -c\circ x^\nu = -1 + \epsilon_0^\nu \ \text{ with } \epsilon^\nu := ||(E^\nu, \epsilon_0^\nu)|| \to 0$$

for $\nu \to \infty$. Here, the element (E^ν, ϵ_0^ν) is to be seen as a vector in $\mathbb{R}^{m^2+1}$. With a strictly feasible $\overline{x}$ we have $A(\overline{x})\succ 0$ and we can choose $M > 0$ large enough such that $M\epsilon^\nu A(\overline{x}) - E^\nu \succeq 0$, $\nu \in \mathbb{N}$. This implies

$$M\epsilon^\nu B - \sum_{i=1}^{N}\big(M\epsilon^\nu \overline{x}_i + x_i^\nu\big)A_i \succeq 0, \quad c\circ\big(M\epsilon^\nu \overline{x} + x^\nu\big) = 1 - \epsilon_0^\nu + M\epsilon^\nu c\circ\overline{x}\ .$$

Dividing by $M\epsilon^\nu$ and using $\epsilon^\nu \to 0$ we obtain

$$B - \sum_{i=1}^{N}\big(\overline{x}_i + \frac{x_i^\nu}{M\epsilon^\nu}\big)A_i \succeq 0, \quad c\circ\big(\overline{x} + \frac{x^\nu}{M\epsilon^\nu}\big) \geq \frac{1}{M\epsilon^\nu} - \frac{1}{M} + c\circ\overline{x} \to \infty\ .$$

The other case can be proven similarly.
(b): In view of Lemma 5 and using (2), to prove the first part of the statement, it is sufficient to show that there exist a solution x, Y of

$$(5)\quad \begin{array}{rcrcl} \sum_{i=1}^{n} x_i A_i\circ V & & & \leq & B\circ V, \quad V \in \mathcal{V}^m \\ & & -Y\circ V & \leq & 0, \quad\ \ V \in \mathcal{V}^m \\ & & Y\circ A_i & = & c_i, \quad\ \ i = 1, \dots, n \\ -\sum_{i=1}^{n} x_i c_i & + & B\circ Y & \leq & 0\ . \end{array}$$

Suppose that this system is not solvable. By Lemma 3 there exist $\alpha^\nu \geq 0$, $V_l^\nu, V_k^\nu \in \mathcal{V}^m$, $\mu_k^\nu, \mu_l^\nu \geq 0$, $k \in K_\nu, l \in L_\nu$, $\lambda_i^\nu \in \mathbb{R}$ such that for $\nu \to \infty$

$$\sum_{l\in L_\nu}\mu_l^\nu \begin{pmatrix} A_1\circ V_l^\nu \\ \vdots \\ A_n\circ V_l^\nu \\ 0 \\ B\circ V_l^\nu \end{pmatrix} + \sum_{k\in K_\nu}\mu_k^\nu \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -V_k^\nu \\ 0 \end{pmatrix} + \sum_{i=1}^{n}\lambda_i^\nu \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_i \\ c_i \end{pmatrix} + \alpha^\nu \begin{pmatrix} -c_1 \\ \vdots \\ -c_n \\ B \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Putting $Y^\nu = \sum_{l\in L_\nu}\mu_l^\nu V_l^\nu$, $S^\nu = \sum_{k\in K_\nu}\mu_k^\nu V_k^\nu$, $x^\nu = -\lambda^\nu$, this is equivalent to

$$(6)\quad \begin{array}{lcl} A_i\circ Y^\nu - \alpha^\nu c_i + \epsilon_i^\nu & = & 0, \qquad i = 1, \dots n, \\ \alpha^\nu B - \sum_{i=1}^{n} x_i^\nu A_i + E^\nu & = & S^\nu \succeq 0, \\ B\circ Y^\nu - c\circ x^\nu & = & -1 + \epsilon_0^\nu\ , \end{array}$$

where $\epsilon^\nu := ||(\epsilon_1^\nu, \dots, \epsilon_n^\nu, E^\nu, \epsilon_0^\nu)|| \to 0$ for $\nu \to \infty$. By defining the numbers $\kappa^\nu := \max\{||(Y^\nu, S^\nu)||, ||x^\nu||, \alpha^\nu\}$ we distinguish between two cases.

Case $\kappa^\nu \leq M$, $\nu \in \mathbb{N}$: Then there exist convergent subsequences $Y^\nu \to Y$, $S^\nu \to S$, $x^\nu \to x$, $\alpha^\nu \to \alpha$ and from (6) we find

$$(7)\quad A_i\circ Y = \alpha c_i,\ i = 1, \dots, n, \quad \alpha B - \sum_{i=1}^{n} x_i A_i = S \succeq 0, \quad B\circ Y - c\circ x = -1\ .$$

If $\alpha > 0$ then by dividing the relations (7) by α we obtain a solution of the system (5), a contradiction. If $\alpha = 0$ then in view of (7) with $\overline{x} \in F_P$, $\overline{Y} \in F_D$ the vectors $x(t) = \overline{x} + tx,\ Y(t) = \overline{Y} + tY$ are feasible with $B \circ Y(t) - c \circ x(t) = B \circ \overline{Y} - c \circ \overline{x} - t \to -\infty$ for $t \to \infty$ contradicting our assumption.

Case $\kappa^\nu \to \infty$, $\nu \to \infty$ (for some subsequence): By dividing (6) by κ^ν and taking converging subsequences we obtain with some $\hat{Y} \succeq 0$, $\hat{S} \succeq 0$, $\hat{\alpha} \geq 0$, $\hat{x}$,

$$(8) \qquad \begin{pmatrix} A_1 \circ \hat{Y} \\ \vdots \\ A_n \circ \hat{Y} \\ -\hat{S} \\ B \circ \hat{Y} \end{pmatrix} - \sum_{i=1}^{n} \hat{x}_i \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_i \\ c_i \end{pmatrix} + \hat{\alpha} \begin{pmatrix} -c_1 \\ \vdots \\ -c_n \\ B \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and $\max\{||(\hat{Y}, \hat{S})||, ||\hat{x}||, \hat{\alpha}\} = 1$. It now follows that $\hat{\alpha} > 0$. In fact for $\hat{\alpha} = 0$, by multiplying (8) with $(-\overline{x}, -\overline{Y}, 1)$, $\overline{x}$, $\overline{Y}$ strict feasible we find using $-A_i \circ \overline{Y} + c_i = 0$

$$(9) \qquad A(\overline{x}) \circ \hat{Y} + \hat{S} \circ \overline{Y} = 0 \quad \text{with} \quad A(\overline{x}), \overline{Y} \succ 0 \,.$$

In view of Lemma 4 it follows $\hat{Y} = \hat{S} = 0$ and by the linear independency of A_i in (8) also $\hat{x} = 0$, a contradiction. The relation $\hat{\alpha} > 0$ implies that (6) is valid with $\alpha^\nu \to \infty$ (some subsequence). Now we can choose Y_ϵ^ν such that with some $M_0 > 0$

$$(10) \qquad A_i \circ Y_\epsilon^\nu = \epsilon_i^\nu,\ i = 1, \ldots, n \quad \text{and} \quad ||Y_\epsilon^\nu|| \leq M_0 \epsilon^\nu \text{ for all } \nu \in \mathbb{N} \,.$$

Thus, with strictly feasible $\overline{x}, \overline{Y}$ there exists $M > 0$ such that

$$(11) \qquad Y_\epsilon^\nu + M\epsilon^\nu \overline{Y} \succeq 0, \quad M\epsilon^\nu \big(B - \sum_{i=1}^{n} \overline{x}_i A_i\big) - E^\nu \succeq 0,\ \nu \in \mathbb{N} \,.$$

For $Y^\nu + Y_\epsilon^\nu + M\epsilon^\nu \overline{Y} \succeq 0$, $x^\nu + M\epsilon^\nu \overline{x}$ we find using (6), (10), (11) and $\epsilon^\nu, \epsilon_0^\nu \to 0$

(12)

$$\begin{array}{lcl} A_i \circ (Y^\nu + Y_\epsilon^\nu + M\epsilon^\nu \overline{Y}) - (\alpha^\nu + M\epsilon^\nu) c_i & = & 0, \qquad i = 1, \ldots n, \\ (\alpha^\nu + M\epsilon^\nu) B - \sum_{i=1}^{n} \big(x_i^\nu + M\epsilon^\nu \overline{x}_i\big) A_i & \succeq & 0, \\ B \circ (Y^\nu + Y_\epsilon^\nu + M\epsilon^\nu \overline{Y}) - c \circ (x^\nu + M\epsilon^\nu \overline{x}) & = & -1 + \epsilon_0^\nu + O(\epsilon^\nu) \leq -\frac{1}{2} \,. \end{array}$$

for any fixed ν large enough. Since $\alpha^\nu \to \infty$ we obtain $\alpha^\nu + M\epsilon^\nu > 0$ for large ν. By dividing (12) by $\alpha^\nu + M\epsilon^\nu > 0$ we have a solution of (5) in contradiction to our assumption. This shows the first part of (b).

The optimality conditions are obtained as follows. Suppose x is a solution of P. As shown, there exists a solution Y of D with $0 = B \circ Y - c \circ x = Y \circ A(x)$. Lemma 4 implies $Y \cdot A(x) = 0$. On the other hand if for $x \in F_P$ the vector $Y \in F_D$ satisfies $Y \cdot A(x) = 0$ and thus $Y \circ A(x) = 0$ then, by Lemma 5, x is a solution of P. The optimality conditions for $Y \in F_D$ are obtained similarly. $\square$

The proof for the semidefinite case is longer than for linear programs. This is because the set

$$S = \text{cone } (S_0^1) + \text{cone } (S_0^2) + \text{lin } (S_1) + \text{cone } \{s_0\}$$

with $S_0^1 = \{(A_1 \circ V, .., A_n \circ V, 0, B \circ V) \mid V \in \mathcal{V}^m\}$, $S_0^2 = \{(0, .., 0, -V, 0) \mid V \in \mathcal{V}^m\}$, $S_1 = \{(0, .., 0, A_i, c_i) \mid i = 1, \ldots, n\}$, $s_0 = (-c_1, .., -c_n, B, 0)$, need not to be closed. This, although the strict feasibility assumptions in Theorem 2(b) imply that the set cone (S_0^1)+cone (S_0^2)+lin (S_1) is closed. Hence, in the proof of Theorem 2(b), the case $\kappa^\nu \to \infty$ cannot be excluded. This complication is not present in linear programming since cones generated by finitely many vectors are always closed.

For further details on semidefinite programming, such as duality gaps, we refer to [4]. Commonly the duality results and optimality conditions for semidefinite problems are obtained by transforming the semidefinite programs into a more abstract cone-constrained form. Our approach avoids such a transformation by transforming the programs into a special case of a semi-infinite problem (see also Section 3).

3. STRONG DUALITY RESULTS IN SEMI-INFINITE PROGRAMMING

In this section we briefly outline how the same approach can be applied to linear semi-infinite programs. A common linear semi-infinite problem is of the form

$$\text{P}: \quad \max_{x \in \mathbb{R}^n} \ c \circ x \quad \text{s.t.} \quad b(t) - \sum_{i=1}^{n} x_i a_i(t) \geq 0 \ , \ \text{for all } t \in T \ ,$$

where $c \in \mathbb{R}^n$ is a given vector and $b(t), a_i(t) \in C(T, \mathbb{R})$, T a compact subset of a topological space. Again we have replaced $c^{\mathrm{T}} x$ by $c \circ x$. $C(T, \mathbb{R})$ denotes the space of real-valued functions f, continuous on T, with norm $||f|| = \max\{|f(t) \mid t \in T\}$. Note, that in view of (2) the semidefinite program in the previous section can be written as a semi-infinite program by defining

$$b(t) = t^{\mathrm{T}} B t, \quad a_i(t) = t^{\mathrm{T}} A_i t, \ i = 1, \ldots, n, \quad t \in T := \{t \in \mathbb{R}^m \mid ||t|| = 1\} \ .$$

For $f \in C(T, \mathbb{R})$ we write $f \geq 0$ ($f > 0$) if $f(t) \geq 0$ ($f(t) > 0$) for all $t \in T$. The dual $C(T, \mathbb{R})^*$ of the space $C(T, \mathbb{R})$ is the space of all real-valued Borel measures y on T (see [3]). We define

$$f \circ y = \int_T f(t) dy(t) \ , \quad f \in C(T, \mathbb{R}), \ y \in C(T, \mathbb{R})^*.$$

The measure y is said to be non-negative (notation $y \geq 0$) if $f \circ y \geq 0$ for all $f \in C(T, \mathbb{R})$, $f \geq 0$ and positive ($y > 0$) if $f \circ y > 0$ for all $f \in C(T, \mathbb{R})$, $f \geq 0$, $f \neq 0$. The dual of P then reads

$$\text{D}: \quad \min_{y \in C(T, \mathbb{R})^*} \ b \circ y \quad \text{s.t.} \quad a_i \circ y = c_i \ , \ i = 1, \ldots, n \ , \quad y \geq 0 \ .$$

As before let v_P, v_D denote the values of P, D and F_P, F_D the feasible sets. Elements $x \in F_P$ and $y \in F_D$ are said to be strictly feasible if

$$a(x) := b - \sum_{i=1}^{n} x_i a_i > 0 \quad \text{and} \quad y > 0 \,.$$

We introduce the set $K_1^+ = \{f \in C(T, \mathbb{R}) \mid f \geq 0, ||f|| \leq 1\}$.

With these settings we can proceed as in the semidefinite case. The full system for the solutions x of P, y of D corresponding to (5), for example, becomes in the semi-infinite case:

$$\begin{array}{lcccll} \sum_{i=1}^{n} x_i a_i(t) & & & \leq & b(t), & t \in T, \\ & & -q \circ y & \leq & 0, & q \in K_1^+, \\ & & a_i \circ y & = & c_i, & i = 1, \ldots, n, \\ -\sum_{i=1}^{n} x_i c_i & + & b \circ y & \leq & 0 \,. & \end{array}$$

By considering some appropriate modifications in the proofs of Section 2 we can prove weak and strong duality results for semi-infinite programs along the same lines as in the semidefinite case. For shortness we only give the strong duality result.

Theorem 3. (Strong Duality) *The following hold.*

(a) *Suppose P is strictly feasible. Then $F_D = \emptyset$ if and only if $v_P = \infty$. Suppose D is strictly feasible. Then $F_P = \emptyset$ if and only if $v_D = -\infty$.*

(b) *Suppose P and D are strictly feasible. Then P and D have solutions x and y satisfying $c \circ x = b \circ y$. Moreover, the following optimality conditions hold*

$$x \in F_P \textit{ solves } P \iff \textit{there exists } y \in F_D \textit{ such that } a(x) \circ y$$
$$y \in F_D \textit{ solves } D \iff \textit{there exists } x \in F_P \textit{ such that } a(x) \circ y = 0 \,.$$

For further details on semi-infinite programming we refer to the paper [2]. In [1] the results of Theorem 2 are obtained in a different way by treating semidefinite programs as a special case of semi-infinite problems.

4. PROOFS OF THE AUXILIARY LEMMATA

For completeness, in this section, the proofs of all auxiliary lemmata of Section 2 will be presented.

Proof of Lemma 3. We prove the statement by using the following standard separation theorem: Let $S \subset \mathbb{R}^q$ be a convex closed set and $y \in \mathbb{R}^q$. Then precisely one of the alternatives (a'), (b') holds,

(a') There exist $\xi \in \mathbb{R}^q$, $\alpha \in \mathbb{R}$ such that $\xi^{\mathrm{T}} s \leq \alpha$, $s \in S$, $\xi^{\mathrm{T}} y > \alpha$,

(b') $y \in S$.

It is easy to show that if (b) is valid then (a) cannot hold. Suppose now that (b) is not true. By putting $y = (0, -1)$, $S := \text{clos}\,\big(\text{cone}\,(S_0) + \text{lin}\,(S_1)\big)$

the condition (b') is not fulfilled. Thus by (a') there exist a vector $(\overline{\xi}, \xi_q) \in \mathbb{R}^q$, $\alpha \in \mathbb{R}$ such that

$$\begin{aligned} \overline{\xi}^{\mathrm{T}} b + \xi_q \beta &\leq \alpha \quad \text{for all } (b,\beta) \in \text{cone } (S_0) \\ \overline{\xi}^{\mathrm{T}} c + \xi_q \gamma &\leq \alpha \quad \text{for all } (c,\gamma) \in \text{lin } (S_1) \\ -\xi_q &> \alpha \; . \end{aligned} \tag{13}$$

With $(c,\gamma) \in \text{lin } (S_1)$, $(b,\beta) \in \text{cone } (S_0)$ these relations also holds for $\pm t(c,\gamma)$, $t(b,\beta)$, $t \geq 0$. This implies $\overline{\xi}^{\mathrm{T}} c + \xi_q \gamma = 0$, $\overline{\xi}^{\mathrm{T}} b + \xi_q \beta \leq 0$ and we can choose $\alpha = 0$. By dividing (13) by $-\xi_q$ we obtain with $\xi = -\overline{\xi}/\xi_q$ the relation $\xi^{\mathrm{T}} b \leq \beta$, $\xi^{\mathrm{T}} c = \gamma$ for all $(b,\beta) \in \text{cone } (S_0)$, $(c,\gamma) \in \text{lin } (S_1)$, i.e. (a). □

Proof of Lemma 4: $A \cdot B = 0$ directly implies $A \circ B = \text{tr } (A \cdot B) = 0$. To prove the converse, consider the transformation of $A, B \succeq 0$ to diagonal form, $A = \sum_{i=1}^n \alpha_i q_i q_i^{\mathrm{T}}$, $B = \sum_{j=1}^n \beta_j v_j v_j^{\mathrm{T}}$, where q_i, v_j are the orthonormal eigenvectors and α_i, β_j the corresponding eigenvalues of A, B. Then with $A \circ B = \text{ tr } (A \cdot B)$ we find using $\alpha_i \beta_j \geq 0$

$$A \circ B = \sum_{i,j=1}^n \alpha_i \beta_j \text{ tr } (q_i q_i^{\mathrm{T}} v_j v_j^{\mathrm{T}}) = \sum_{i,j=1}^n \alpha_i \beta_j \; (v_j^{\mathrm{T}} q_i q_i^{\mathrm{T}} v_j) = \sum_{i,j=1}^n \alpha_i \beta_j \; (q_i^{\mathrm{T}} v_j)^2 \geq 0 \; .$$

Moreover, $A \circ B = 0$ implies $\alpha_i \beta_j (q_i^{\mathrm{T}} v_j)^2 = 0$ or $\alpha_i \beta_j (q_i^{\mathrm{T}} v_j) = 0$ for all i, j and then

$$A \cdot B = \sum_{i,j=1}^n \alpha_i \beta_j \; q_i q_i^{\mathrm{T}} v_j v_j^{\mathrm{T}} = \sum_{i,j=1}^n \alpha_i \beta_j \; (q_i^{\mathrm{T}} v_j) \; q_i v_j^{\mathrm{T}} = 0 \; .$$

When $A \succ 0$ then in particular, the matrix A is regular and $A \cdot B = 0$ implies $B = A^{-1} 0 = 0$. □

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