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A CONDITION FOR UNIVALENCY

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Abstract. In this paper we establish a very simple and useful univalence criteria for a class of functions defined by an integral operator.

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1. INTRODUCTION

We denote by $U_r = \{z \in C : |z| < r\}$ the disk of z-plane, where $r \in (0, 1]$, $U_1 = U$ and $I = [0, \infty)$. Let A be the class of functions f analytic in U such that $f(0) = 0$, $f'(0) = 1$. Our consideration are based on the theory of Löwner chains; we first recall here the basic result of this theory, from Pommerenke.

THEOREM 1.1. ([2]). *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$, $a_1(t) \neq 0$ be analytic in U_r for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r. For almost all $t \in I$ suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z,t)$ is analytic in U and satisfies the condition $Re p(z,t) > 0$ for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$ the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

2. MAIN RESULTS

THEOREM 2.1. *Let α, β be real numbers, $\alpha > 0$, $\beta \geq 1$, and let $f \in A$. If the inequalities*

$$\left| \frac{zf'(z)}{f(z)} - \beta \right| < \beta, \tag{1}$$

$$\left| \frac{1}{\beta}\left(\frac{zf'(z)}{f(z)} - \beta\right)|z|^{2(\alpha+\beta-1)} + \frac{1-|z|^{2(\alpha+\beta-1)}}{\alpha+\beta-1}\left(\frac{zf'(z)}{f(z)} - \beta\right) \right| \leq 1 \tag{2}$$

are true for all $z \in U$, then the function

$$F(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u)\mathrm{d}u\right)^{1/\alpha} \tag{3}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. Let us prove that there exists $r \in (0,1]$ such that the function $L : U_r \times I \to C$ defined as

$$L(z,t) = \Bigg[(\alpha+\beta-1)\int_0^{e^{-t}z} u^{\alpha-1} f'(u)\mathrm{d}u$$

$$+\beta(e^{(\alpha+2\beta-2)t} - e^{-\alpha t})z^{\alpha} \cdot \frac{f(e^{-t}z)}{e^{-t}z}\Bigg]^{1/\alpha} \tag{4}$$

is analytic in U_r for all $t \in I$. Denoting

$$g_1(z,t) = (\alpha+\beta-1)\int_0^{e^{-t}z} u^{\alpha-1} f'(u)\mathrm{d}u$$

we have $g_1(z,t) = z^{\alpha} g_2(z,t)$ and it is easy to see that g_2 is analytic in U for all $t \in I$ and $g_2(0,t) = (\alpha+\beta-1)/\alpha \cdot e^{-\alpha t}$. The function

$$g_3(z,t) = g_2(z,t) + \beta(e^{(\alpha+2\beta-2)t} - e^{-\alpha t})\frac{f(e^{-t}z)}{e^{-t}z}$$

is also analytic in U and

$$g_3(0,t) = \frac{(\alpha-1)(1-\beta)}{\alpha} e^{-\alpha t} + \beta e^{(\alpha+2\beta-2)t}.$$

Let us now prove that $g_3(0,t) \neq 0$ for any $t \in I$. We have $g_3(0,0) = (\alpha+\beta-1)/\alpha$ and from the hypothesis $\alpha+\beta-1>0$. Assume that there exists $t_0 > 0$ such that $g_3(0,t_0) = 0$. Then $e^{2(\alpha+\beta-1)t_0} = (\alpha-1)(\beta-1)/(\alpha\beta)$. In view of $\alpha+\beta-1>0$ it follows $e^{2(\alpha+\beta-1)t_0} > 1$ and $(\alpha-1)(\beta-1)/(\alpha\beta) < 1$ and then we conclude that $g_3(0,t) \neq 0$ for all $t \in I$. Therefore, there is a disk $U_{r_1},\ 0 < r_1 \leq 1$, in which $g_3(z,t) \neq 0$ for all $t \in I$.Then we choose the uniform branch of $(g_3(z,t))^{1/\alpha}$ analytic in U_{r_1} , denoted by $g(z,t)$, that is equal to

$$a_1(t) = e^{\frac{\alpha+2\beta-2}{\alpha}t}\left[\beta + \frac{(\alpha-1)(1-\beta)}{\alpha}\cdot e^{-2(\alpha+\beta-1)t}\right]^{1/\alpha} \tag{5}$$

at the origin. From these considerations, it results that the relation (4) may be written as

$$L(z,t) = zg(z,t) = a_1(t)z + a_2(t)z^2 + \ldots,$$

where $a_1(t)$ is given by (5). Because $\alpha > 0,\ \beta \geq 1$, then $\alpha+2\beta-2>0$ and $\lim_{t\to\infty}|a_1(t)| = \infty$. Moreover, $a_1(t) \neq 0$ for all $t \in I$. From the analyticity of $L(z,t)$ in U_{r_1}, it follows that there is a number r_2, $0 < r_2 < r_1$, and a constant $K = K(r_2)$ such that

$$|\ L(z,t)/a_1(t)\ | \ <\ K, \qquad \forall z \in U_{r_2}, \quad t \in I.$$

and then$\{\ L(z,t)/a_1(t)\ \}$ is a normal family in U_{r_2}. From the analyticity of $\partial L(z,t)/\partial t$, for all fixed numbers $T > 0$ and $r_3,\ 0 < r_3 < r_2$, there exists a constant $K_1 > 0$ (that depends on T and r_3) such that

$$\left| \frac{\partial L(z,t)}{\partial t} \right| < K_1, \quad \forall z \in U_{r_3}, \quad t \in [0,T].$$

Therefore the function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_3}. Let us set

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t} \quad \text{and} \quad w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}\,.$$

The function $p(z,t)$ is analytic in U_r, $0 < r < r_3$ and so is $w(z,t)$. The function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$, if the function $w(z,t)$ can be continued analytically in U and $|w(z,t)| < 1$ for all $z \in U$and $t \in I$. After computation we obtain

$$w(z,t) = \frac{1}{\beta}\left(\frac{\mathrm{e}^{-t}zf'(\mathrm{e}^{-t}z)}{f(\mathrm{e}^{-t}z)} - \beta\right)\cdot \mathrm{e}^{-2(\alpha+\beta-1)t} + \frac{1-\mathrm{e}^{-2(\alpha+\beta-1)t}}{\alpha+\beta-1}\left(\frac{\mathrm{e}^{-t}zf'(\mathrm{e}^{-t}z)}{f(\mathrm{e}^{-t}z)} - \beta\right). \tag{6}$$

From (1) and (2) we deduce that the function $w(z,t)$ is analytic in the unit disk U. In view of (1), from (6) we have

$$|w(z,0)| = \left|\frac{1}{\beta}\left(\frac{zf'(z)}{f(z)} - \beta\right)\right| < 1\,. \tag{7}$$

For $z = 0,\ t > 0$, in view of $\alpha > 0,\ \beta \geq 1$, from (6) we obtain

$$|w(0,t)| = \frac{\beta-1}{\beta(\alpha+\beta-1)}\cdot\left|\beta + (\alpha-1)\mathrm{e}^{-2(\alpha+\beta-1)t}\right| < 1\,. \tag{8}$$

If $t > 0$ is a fixed number and $z \in U,\ z \neq 0$, then the function $w(z,t)$ is analytic in $\bar{U}$ because $|\mathrm{e}^{-t}z| \leq \mathrm{e}^{-t} < 1$ for all $z \in \bar{U}$ and it is known that

$$|w(z,t)| = \max_{|\zeta|=1}|w(\zeta,t)| = |w(\mathrm{e}^{i\theta},t)|, \quad \theta = \theta(t) \in R\,. \tag{9}$$

Let us denote $u = \mathrm{e}^{-t}\mathrm{e}^{i\theta}$. Then $|u| = \mathrm{e}^{-t}$ and from (6) we get

$$|w(\mathrm{e}^{i\theta},t)| = \left|\frac{1}{\beta}\left(\frac{uf'(u)}{f(u)} - \beta\right)|u|^{2(\alpha+\beta-1)} + \frac{1-|u|^{2(\alpha+\beta-1)}}{\alpha+\beta-1}\left(\frac{uf'(u)}{f(u)} - \beta\right)\right|.$$

Because $u \in U$, the relation (2) implies $|w(\mathrm{e}^{i\theta},t)| \leq 1$ and from (7), (8) and (9) we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$. From Theorem 1.1 it follows that the function $L(z,t)$ has an analytic and univalent extension to the whole disk U, for each $t \in I$. For $t = 0$ it follows that the function

$$L(z,0) = \left((\alpha+\beta-1)\int_0^z u^{\alpha-1}f'(u)\mathrm{d}u \right)^{1/\alpha}$$

is analytic and univalent in U and then the function F defined by (3) is also analytic and univalent in U.

REMARK. If we ask for α to be $\alpha \geq 1$, then if the inequality (1) is true, it results that the inequality (2) is also true and we have the following results:

THEOREM 2.2. *Let α, β be real numbers , $\alpha \geq 1$, $\beta \geq 1$ and let $f \in A$. If the inequality*

$$\left| \frac{zf'(z)}{f(z)} - \beta \right| < \beta \tag{1}$$

is true for all $z \in U$, then the function

$$F((z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u) \mathrm{d}u \right)^{1/\alpha} \tag{3}$$

is analytic and univalent in U.

Proof. Since $\alpha \geq 1$, the left-hand side of the inequality (2) can be majorated and in view of (1) we obtain:

$$\left| \frac{zf'(z)}{f(z)} - \beta \right| \cdot \left| \frac{1}{\beta} |z|^{2(\alpha+\beta-1)} + \frac{1 - |z|^{2(\alpha+\beta-1)}}{\alpha+\beta-1} \right|$$

$$\leq \left| \frac{zf'(z)}{f(z)} - \beta \right| \cdot \left(\frac{1}{\beta} |z|^{2(\alpha+\beta-1)} + \frac{1 - |z|^{2(\alpha+\beta-1)}}{\beta} \right) = \frac{1}{\beta} \left| \frac{zf'(z)}{f(z)} - \beta \right| < 1.$$

Then (2) is satisfied and from Theorem 2.1, the function F defined by (3) is analytic and univalent in U.

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