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EQUILIBRIUM PROBLEMS AND VARIATIONAL INEQUALITIES

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Abstract. In this paper, we suggest and analyze some iterative methods for solving equilibrium problems with trifunction by using the auxiliary principle technique. We prove that the convergence of the proposed methods either requires only pseudomonotonicity or partially relaxed strongly monotonicity. We also consider the concept of well-posedness for equilibrium problems with trifunction and obtain some new results. It is shown that the auxiliary principle technique developed in this paper can be extended for regularized equilibrium problems with some minor modifications. Since equilibrium problems with trifunction include the classical equilibrium problems, variational inequalities and complementarity problems as special cases, results proved in this paper continue to hold for these problems. Our results can be viewed as a novel application of the auxiliary principle technique.

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Key words. Equilibrium problems, variational inequalities, auxiliary principle, proximal methods, convergence, well-posedness.

1. INTRODUCTION

Equilibrium problems theory provides us with a unified, natural, innovative and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation, elasticity and optimization. This theory has witnessed an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences. Several generalizations and extensions of the equilibrium problems have been considered and investigated in several direction. A significant and useful generalization of the equilibrium problems is called the equilibrium problems with trifunction, which was introduced and investigated by Noor and Oettli [20] in the setting of topological spaces. Blum and Oettli [2] and Noor and Oettli [20] have shown that the equilibrium problems include several nonlinear programming, complementarity, fixed-point, Nash equilibrium, transportation, network problems and variational inequalities as special cases. In recent years, several numerical methods including projection, Wiener-Hopf (resolvent) equations, auxiliary principle technique have been developed for variational inequalities and related optimization problems. Unfortunately, the projection method and its variant forms including the Wiener-Hopf equations can not be extended for solving equilibrium problems, since it is not possible to find the projection of the bifunction (trifunction) from the whole space onto the convex set. To overcome this drawback, one usually uses the auxiliary principle technique. Glowinski, Lions and Tremolieres [7] has used this technique to study the existence of a solution of mixed variational inequalities, whereas Noor [16], [17], [18] has used this technique to suggest and analyze a number of iterative methods for solving various classes classes of variational inequalities and equilibrium problems. In this paper, we again use the auxiliary principle technique to suggest and analyze some iterative methods for equilibrium problems with trifunction. We have studied the convergence criteria of these methods under some mild conditions. As a consequence of this approach, we construct the gap (merit) function for equilibrium problems, which can be used to develop descent-type methods for solving equilibrium problems. We also introduce the concept of well-posedness for equilibrium problems and obtain some results. We note that almost all the results obtained so far have been obtained in the setting of the convexity. We also consider the equilibrium problems with trifunction in the setting of uniformly prox-regular convex sets, which are not convex sets and show that the auxiliary principle technique can be extended for solving the regularized equilibrium problems. Note that the regularized equilibrium problems include the equilibrium problems and variational inequalities as special cases. Regular equilibrium problems are new problems in the setting of uniformly prox-regular sets, which are not convex sets. The interested reader is urged to explore these problems further and discover some new, novel and innovative applications of the reqularized equilibrium problems in the setting of different normed space. Our results can be viewed as significant extension and generalization of the previously known results for solving classical variational inequalities and equilibrium problems.

2. PRELIMINARIES

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let $T : H \to H$ be a nonlinear operator. For a given nonlinear function $F(.,.,.) : K \times K \times K \to R$, consider the problem of finding $u \in K$ such that

$$F(u, Tu, v) \geq 0, \quad \forall\, v \in K, \tag{1}$$

which is called the *equilibrium problem with trifunction*, considered and investigated by Noor and Oettli [20] in 1994. If $F(u.Tu, v) \equiv F(u, v)$, then we obtain the classical equilibrium problem considered by Blum and Oettli [2] and Noor and Oettli [20]. For applications and numerical results of equilibrium problems, see [2], [12], [17], [18], [20].

If $F(u, Tu, v) = \langle Tu, \eta(v, u)\rangle$, where $\eta(.,.) : H \times H \to H$ is a single-valued mapping, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, \eta(v, u)\rangle \geq 0, \quad \forall\, v \in K, \tag{2}$$

which is called the variational-like inequality, see Noor [17]. Here the set K is invex set, which may not be a convex set. It is well-known that variational-like inequality problems are closely related to the preinvex functions, which are not necessarily convex functions, see [17].

If $F(u, Tu, v) \equiv \langle Tu, g(v) - g(u) \rangle$, where $g, T : H \to H$ are nonlinear single-valued operators, then problem (1) is equivalent to finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall \quad v \in H : g(v) \in K, \tag{3}$$

is called the *general variational inequality.* General variational inequalities were introduced by Noor in 1988. It has been shown that a wide class of nonsymmetric, odd-order free, moving, equilibrium and optimization problems can be studied by the general variational inequalities, see [13], [14], [17], [18], [19].

If $F(u, Tu, v) = \langle Tu, v - u \rangle$, where $T : H \to H$ is a nonlinear operator, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall \, v \in K, \tag{4}$$

which is known as the classical variational inequality, introduced and studied by Stampacchia [23] in 1964. It is well-known that a wide class of obstacle, unilateral, contact, free, moving and equilibrium problems arising in mathematical, engineering, economics and finance can be studied in the unified and general framework of the variational inequalities of type (4). For the physical and mathematical formulation of problems (1)–(4), see [1], [2], [4]–[21], [23]–[25] and the references therein.

We also need the following concepts and results.

Lemma 1. $\forall u, v \in H$,

$$2\langle u, v \rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2. \tag{5}$$

Definition 1. The trifunction $F(.,.,.) : K \times K \times K \to R$ with respect to the operator T is said to be:

(i) *pseudomonotone*, if

$$F(u, Tu, v) \geq 0 \quad \Longrightarrow -F(v, Tv, u) \geq 0, \quad \forall u, v \in K.$$

(ii) *partially relaxed strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$F(u, Tu, v) + F(v, Tv, z) \leq \alpha \|z - u\|^2, \quad \forall u, v, z \in K.$$

(iii) *hemicontinuous*, if $\forall u, v \in K$, the mapping $t \in [0, 1]$ implies that $F(u + t(v - u), T(u + t(v - u)), v)$ is continuous.

Note that for $z = u$, partially relaxed strongly monotonicity reduces to

$$F(u, Tu, v) + F(v, Tv, u) \leq 0, \quad \forall u, v \in K,$$

which is known as the monotonicity of $F(.,.)$. It is known [5] that monotonicity implies pseudomonotonicity, but the converse is not true.

3. ITERATIVE SCHEMES

We suggest and analyze some proximal methods for equilibrium problems (1) using the auxiliary principle technique of Glowinski, Lions and Tremolieres [7] as developed by Noor and Noor [18].

For a given $u \in K$, consider the auxiliary problem of finding a unique $w \in K$ such that

$$\rho F(w, Tw, v) + \langle w - u + \gamma(u - u), v - w\rangle \geq 0, \quad \forall\, v \in K, \tag{6}$$

where $\rho > 0$ and $\gamma > 0$ are constants.

We note that if $w = u$, then clearly w is solution of the equilibrium problem (1). This observation enables us to suggest and analyze the following iterative method for solving (1).

Algorithm 1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_{n+1}, Tu_{n+1}, v) + \langle u_{n+1} - u_n + \gamma_n(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \ \forall\, v \in K,$$

which is known as the inertial proximal method for solving equilibrium problem with trifunction (1). Such type inertial proximal methods have been considered by Alvarez and Attouch [1], Noor and Noor [18], Moudafi [12] and Noor [17] for solving variational inequalities and equilibrium problems.

For $\gamma_n = 0$, Algorithm 1 collapses to:

Algorithm 2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_{n+1}, Tu_{n+1}, v) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall\, v \in K, \tag{7}$$

which is called the proximal method for solving problem (1). This shows that the inertial proximal methods include the classical proximal methods as a special case.

If $F(u, Tu, v) = \langle Tu, v - u\rangle$, where $T : K \to H$ is a nonlinear continuous operator, then Algorithm 1 reduces to:

Algorithm 3. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_{n+1} + u_{n+1} - u_n + \gamma_n(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall\, v \in K,$$

which can be written as

$$u_{n+1} = P_K[u_n - \rho Tu_{n+1} + \gamma_n(u_n - u_{n-1})], \quad n = 0, 1, 2, \ldots,$$

where P_K is the projection of H onto the convex set K. Algorithm 2 is known as the inertial proximal point algorithm for solving variational inequalities and has been studied by Noor [17]. In a similar way, one can obtain several iterative methods for variational-like inequalities (2), general variational inequalities (3) and their special cases, see [16], [17].

We now study the convergence analysis of Algorithm 2. The analysis is in the spirit of Noor [1], [17]. The convergence analysis of Algorithms 1, and 3 can be studied in a similar way.

THEOREM 1. *Let $\bar{u} \in K$ be a solution of* (1) *and let u_{n+1} be the approximate solution obtained from* Algorithm 2. *If the trifunction $F(.,.,.)$ is pseudomonotone, then*

$$\|u_{n+1}-\bar{u}\|^2 \leq \|u_n-\bar{u}\|^2 - \|u_{n+1}-u_n\|^2. \tag{8}$$

Proof. Let $\bar{u} \in K$ be a solution of (1). Then $F(\bar{u}, T\bar{u}, v) \geq 0$, $\forall\, v \in K$, which implies that

$$-F(v, Tv, \bar{u}) \geq 0, \quad \forall\, v \in K, \tag{9}$$

since $F(.,.,.)$ is pseudomonotone.

Taking $v = u_{n+1}$ in (9), we have

$$-F(u_{n+1}, Tu_{n+1}, \bar{u}) \geq 0. \tag{10}$$

Now taking $v = \bar{u}$ in (7), we obtain

$$\rho F(u_{n+1}, Tu_{n+1}, \bar{u}) + \langle u_{n+1}-u_n, \bar{u}-u_{n+1}\rangle \geq 0. \tag{11}$$

From (10) and (11), we have

$$\langle u_{n+1}-u_n, \bar{u}-u_{n+1}\rangle \geq -\rho F(u_{n+1}, Tu_{n+1}, \bar{u}) \geq 0. \tag{12}$$

Setting $u = \bar{u} - u_{n+1}$ and $v = u_{n+1} - u_n$ in (5), we obtain

$$\begin{aligned} 2\langle u_{n+1}-u_n, \bar{u}-u_{n+1}\rangle &= \|\bar{u}-u_n\|^2 \\ &\quad -\|\bar{u}-u_{n+1}\|^2 - \|u_n-u_{n+1}\|^2. \end{aligned} \tag{13}$$

Combining (12) and (13), we have

$$\|u_{n+1}-\bar{u}\|^2 \leq \|u_n-\bar{u}\|^2 - \|u_{n+1}-u_n\|^2,$$

the required result. □

THEOREM 2. *Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from* Algorithm 2 *and $\bar{u} \in K$ is a solution of* (1), *then $\lim_{n\to\infty} u_n = \bar{u}$.*

Proof. Let $\bar{u} \in K$ be a solution of (1). From (8), it follows that the sequence $\{\|\bar{u}-u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (8), we have

$$\sum_{n=0}^{\infty} \|u_{n+1}-u_n\|^2 \leq \|u_0-\bar{u}\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1}-u_n\| = 0. \tag{14}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (6) and taking the limit

$n_j \to \infty$ and using (14), we have $F(\hat{u}, T\hat{u}, v) \geq 0$, $\forall\, v \in K$, which implies that $\hat{u}$ solves the equilibrium problem (1) and $\|u_{n+1} - u_n\|^2 \leq \|u_n - \bar{u}\|^2$. Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} u_n = \hat{u}$, the required result. $\square$

It is known that in order to implement the inertial proximal and proximal algorithms, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we suggest another iterative method for solving equilibrium problem (1).

For a given $u \in K$, consider the auxiliary problem of finding a unique $w \in K$ such that

$$\rho F(u, Tu, v) + \langle w - u, v - w\rangle \geq 0, \quad \forall\, v \in K, \tag{15}$$

where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is solution of the equilibrium problem (1). Note that problems (6) and (15) are quite different. In fact, problem (15) is equivalent to an optimization problem, This observation enables us to suggest and analyze the following iterative method for solving equilibrium problem (1).

Algorithm 4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_n, Tu_n, v) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall\, v \in K. \tag{16}$$

If $F(u, Tu, v) \equiv \langle Tu, v - u\rangle$, then Algorithm 4 *is equivalent to the following iterative method for solving variational inequalities* (4).

Algorithm 5. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_n + u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall\, v \in K,$$

or equivalently

$$u_{n+1} = P_K[u_n - \rho Tu_n], \quad n = 0, 1, 2, \ldots$$

where P_K is the projection operator. Algorithm 5 has been studied extensively, see [13], [14], [16], [17], [18], [19], [21], [25]. For suitable and appropriate choice of the function $F(.,.,.)$ and the space H, one can obtain several iterative schemes for solving problems (1)–(4) and related optimization problems.

We now study the convergence analysis of Algorithm 4.

Theorem 3. *Let $\bar{u} \in K$ be a solution of* (1) *and u_{n+1} be the approximate solution obtained from* Algorithm 4. *If $F(.,.,.) : K \times K \times K \to R$ is partially strongly monotone with constant $\alpha > 0$, then*

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\alpha\rho)\|u_{n+1} - u_n\|^2. \tag{17}$$

Proof. Let $\bar{u} \in K$ be a solution of (1). Then

$$F(\bar{u}, T\bar{u}, v) \geq 0, \quad \forall\, v \in K, \tag{18}$$

Taking $v = u_{n+1}$ in (18), we have

$$F(\bar{u}, T\bar{u}, u_{n+1}) \geq 0. \tag{19}$$

Now taking $v = \bar{u}$ in (16), we obtain

$$\rho F(u_n, Tu_n, \bar{u}) + \langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle \geq 0. \tag{20}$$

From (19) and (20), we have

$$\begin{aligned} \langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle &\geq -\rho\{F(u_n, Tu_n, \bar{u}) + F(\bar{u}, T\bar{u}, u_{n+1})\} \\ &\geq -\alpha\rho\|u_n - u_{n+1}\|^2, \end{aligned} \tag{21}$$

since $F(.,.,.)$ is partially relaxed strongly monotone with a constant $\alpha > 0$.

Combining (13) and (21), we have

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\rho\alpha)\|u_{n+1} - u_n\|^2,$$

the required result. □

THEOREM 4. *Let H be a finite dimensional space and let $0 < \rho < \frac{1}{2\alpha}$. If u_{n+1} is the approximate solution obtained from* Algorithm 4 *and $\bar{u} \in H$ is a solution of* (1)*, then* $\lim_{n\to\infty} u_n = \bar{u}$.

Proof. Its proof is similar to Theorem 2. □

It is obvious that the auxiliary equilibrium problem (16) is equivalent to finding the minimum of the functional $I[w]$ over the convex set K, where

$$I[w] = (1/2)\langle w - u, w - u\rangle - \rho F(u, Tu, w), \tag{22}$$

which is known as the auxiliary energy (virtual work, potential) function associated with the problem (16). Using this functional $I[w]$, one can reformulate the equilibrium problem (1) as an equivalent optimization problem:

$$\Psi_\alpha(u) = \max_{w\in K}\{-\rho F(u, Tu, w) - (\alpha/2)\|u - w\|^2\}, \tag{23}$$

where $\alpha > 0$ is a constant. Function of the type $\Psi(u)$ defined by (23) is called the regular gap function for the equilibrium problem. Note that for $\alpha = 0$, and $F(u, Tu, v) \equiv \langle Tu, v - u\rangle$, we obtain the original gap function for the variational inequality (4), which is due to Fukushima [4]. From the above discussion and observation, it is clear that can obtain the gap (merit) function for the equilibrium problems (1) by using the auxiliary principle technique. In passing, we remark this is observation is due to Noor [17], where it has been shown that the auxiliary principle technique can be used to construct gap functions for several variational inequalities. This equivalent optimization formulation of the equilibrium problems can be used to develop some descent-type algorithms for solving equilibrium problems under suitable conditions on the function $F(.,.,.)$ by using the technique of Fukushima [4].

4. WELL-POSED EQUILIBRIUM PROBLEMS

In recent years, much attention has been given to introduce the concept of well-posedness for variational of variational inequalities, see [8], [10], [11], [15] and the references therein. In this section, we introduce the similar concepts of well-posedness for equilibrium problems of type (1). The results obtained can be considered as a natural generalization of previous results of Luccheti and Patrone [10], [11], Goeleven and Mantague [8] and Noor [15]. For this purpose, we define the following:

For a given $\epsilon > 0$, we consider the sets

$$A(\epsilon) = \{u \in K : F(u, Tu, v) \geq -\epsilon\|v - u\|, \quad \forall\, v \in K\},$$

$$B(\epsilon) = \{u \in K : F(v, Tv, u) \leq \epsilon\|v - u\|, \quad \forall\, v \in K\}.$$

For a nonempty set $X \subset H$, we define the diameter of X, denoted by $D(X)$, as

$$D(X) = \sup\{\|v - u\|; \quad \forall u, v \in X.$$

Definition 2. We say that the equilibrium problem (1) is *well-posed*, if and only if

$$A(\epsilon) \neq \phi \quad \text{and} \quad D(A(\epsilon)) \to 0, \text{ as } \quad \epsilon \to 0.$$

For $F(u, Tu, v) = \langle Tu, v - u\rangle$, our definition of well-posedness is exactly the same as one introduced by Luccheti and Patrone [10], [11] for variational inequalities and extended by Noor [15] and Goeleven and Mantague [8] for variational-like inequalities and hemivariational inequalities respectively.

Theorem 5. *Let the function $F(.,.,.)$ be pseudomonotone, hemicontinuous and convex in the third argument. Then $A(\epsilon) = B(\epsilon)$.*

Proof. Let $u \in K$ be such that $F(u, Tu, v) \geq -\epsilon\|v - u\|$, $\forall\, v \in K$, which implies that

$$F(v, Tv, u) \leq \epsilon\|v - u\|, \quad \forall\, v \in K, \tag{24}$$

since $F(.,.,.)$ is pseudomonotone.

Thus

$$A(\epsilon) \subset B(\epsilon). \tag{25}$$

Conversely, let $u \in K$ such that (24) hold. Since K is a convex set, $\forall u, v \in K$, $t \in [0, 1]$, $v_t = u + t(v - u) \equiv (1 - t)u + tv \in K$.

Taking $v = v_t$ in (24), we have

$$F(v_t, Tv_t, u) \leq t\epsilon\|v - u\|. \tag{26}$$

Also

$$\begin{aligned} 0 &= F(v_t, Tv_t, v_t) \\ &\leq tF(v_t, Tv_t, v) + (1 - t)F(v_t, Tv_t, u) \\ &\leq tF(v_t, Tv_t, v) + (1 - t)t\epsilon\|v - u\|, \end{aligned}$$

where we have used (26).

Dividing the above inequality by t and letting $t \to 0$, we have

$$F(u, Tu, v) \geq -\epsilon\|v - u\|, \quad \forall\, v \in K,$$

which implies that

$$B(\epsilon) \subset A(\epsilon). \tag{27}$$

Thus from (25) and (27), we have $A(\epsilon) = B(\epsilon)$, the required result. □

Theorem 6. *The set $B(\epsilon)$ is closed under the assumptions of* Theorem 5.

Proof. Let $\{u_n : n \in N\} \subset B(\epsilon)$ be such that $u_n \to u$ in K as $n \to \infty$. This implies that $u_n \in K$ and

$$F(v, Tv, u_n) \leq \epsilon\|v - u_n\|, \quad \forall\, v \in K.$$

Taking the limit in the above inequality as $n \to \infty$, we have

$$F(v, Tv, u) \leq \epsilon\|v - u\|, \quad \forall\, v \in K,$$

which implies that $u \in K$, since K is a closed and convex set. Consequently, it follows that the set $B(\epsilon)$ is closed. □

Using essentially the technique of Goeleven and Mantague [8], we can prove the following results. To convey an idea and for the sake of completeness, we include their proofs.

Theorem 7. *Let $F(.,.,.)$ be pseudomonotone and hemicontinuous. If the equilibrium problem* (1) *is well-posed, then equilibrium problem* (1) *has a unique solution.*

Proof. Let us define the sequence $\{u_k : k \in N\}$ by $u_k \in A(1/k)$. Let $\epsilon > 0$ be sufficiently small and let $m, n \in N$ such that $n \geq m \geq \frac{1}{\epsilon}$. Then $A(\frac{1}{n}) \subset A(\frac{1}{m}) \subset A(\epsilon)$. Thus $\|u_n - u_m\| \leq D(A(\frac{1}{n}))$, which implies that the sequence $\{u_n\}$ is a Cauchy sequence and it converges, that is, $u_k \to u$ in K. From Theorems 5 and 6, we know that the set $A(\epsilon)$ is a closed set. Thus $u \in \cup_{\epsilon>0} A(\epsilon)$, so that u is solution of the equilibrium problem (1). From the second condition of well-posedness, we see that the solution of the equilibrium problem (1) is unique. □

Theorem 8. *Let $F(.,.,.)$ be pseudomonotone and hemicontinuous. If we have $A(\epsilon) \neq 0$, $\forall \epsilon > 0$ and $A(\epsilon)$ is bounded for some ϵ_0, then the equilibrium problem* (1) *has at least one solution.*

Proof. Let $u_n \in A(1/n)$. Then $A(1/n) \subset A(\epsilon)$, for n large enough. Thus for some subsequence $u_n \to u \in K$, we have

$$\begin{aligned} F(v, Tv, u_n) &\leq \frac{1}{n}\|v - u_n\| \\ &\leq \frac{1}{n}\{\|v\| + c\}, \quad \forall\, v \in K. \end{aligned}$$

Taking the limit as $n \to \infty$, we have $F(v, Tv, u) \leq 0$, which implies that $u \in B(0) = A(0)$, by Theorem 5. This shows that $u \in A(0)$, from which it follows that the equilibrium problem (1) has at least one solution. $\square$

REMARK 1. **I.** If the equilibrium problem (1) has a unique solution, then it is clear that $A(\epsilon) \neq 0, \forall \epsilon > 0$ and $\cap_{\epsilon>0} A(\epsilon) = \{u_0\}$.
II. It is known [11] that if the variational inequality (4) has a unique solution, then it is not well-posed.
III. From Theorem 7, we conclude that the unique solution of the equilibrium problem (1) can be computed by using the ϵ-equilibrium problem, that is, find $u_\epsilon \in K$ such that $F(u_\epsilon, T\epsilon, , v) \geq -\epsilon \|v - u_\epsilon\|$, $v \in K$.

5. EXTENSIONS

We would like to point out that the techniques and ideas of Section 3 can be extended for solving the uniformly regularized equilibrium problems, which are defined over the uniformly prox-regular sets K in H. It is known [3], [22] that the uniformly prox-regular sets are nonconvex and include the convex sets as a special case. For this purpose, we need the following concepts from nonsmooth analysis, see [3], [22].

DEFINITION 3. The proximal normal cone of K at u is given by

$$N^P(K; u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in S : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N^P(K; u)$ has the following characterization.

LEMMA 2. *Let K be a closed subset in H. Then $\zeta \in N^P(K; u)$ if and only if there exists a constant $\alpha > 0$ such that*

$$\langle \zeta, v - u \rangle \leq \alpha \|v - u\|^2, \quad \forall\, v \in K.$$

DEFINITION 4. The Clarke normal cone, denoted by $N^C(K; u)$, is defined as

$$N^C(K; u) = \overline{co}[N^P(K; u)],$$

where $\overline{co}$ means the closure of the convex hull.

Clearly $N^P(K; u) \subset N^C(K; u)$, but the converse is not true. Note that $N^P(K; u)$ is always closed and convex, whereas $N^P(K; u)$ is convex, but may not be closed, see [22]. Poliquin et al [22] and Clarke et al [3] have introduced and studied a new class of nonconvex sets, which are also called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. In particular, we have

DEFINITION 5. For a given $r \in (0, \infty]$, a subset K is said to be uniformly r-prox-regular if and only if every nonzero proximal normal to K can be realized by an r-ball, that is, $\forall u \in K$ and $0 \neq \xi \in N^P(K; u)$ with $\|\xi\| = 1$, one has

$$\langle (\xi)/\|\xi\|, v - u\rangle \leq (1/2r)\|v - u\|^2, \quad \forall\, v \in K.$$

It is clear that the class of uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets (see [12], [17], [18]). It is clear that if $r = \infty$, then uniform r-prox-regularity of K is equivalent to the convexity of K. This fact plays an important part in this paper.

It is known that if K is a uniformly r-prox-regular set, then the proximal normal cone $N^P(K; u)$ is closed as a set-valued mapping. Thus, we have $N^C(K; u) = N^P(K; u)$.

We consider the problem of finding $u \in K$ such that

$$F(u, Tu, v) + (1/2r)\|v - u\|^2 \geq 0, \quad \forall\, v \in K, \tag{28}$$

Problem of the type (28) is called the uniformly regularized equilibrium problem. Note that if $r = \infty$, then the uniformly prox-regular set K becomes the convex set K. Consequently problem (28) is exactly the equilibrium problem (1). Using essentially the technique of Section 3, one can suggest and analyze similar iterative schemes for solving uniform regularized equilibrium problems (28) with minor modification and adjustments.

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