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SUBORDINATION CHAINS AND SOLUTIONS OF THE LOEWNER DIFFERENTIAL EQUATION IN $\mathbb{C}^n$

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Abstract. In this paper we continue the work begun in [8] and study the general solution of the Loewner differential equation on the unit ball in $\mathbb{C}^n$. We generalize to several variables a result of Becker concerning the form of arbitrary solutions to the Loewner differential equation. We do not require the solutions to be normalized. In particular, we determine the form of biholomorphic solutions, which need not be unique in higher dimensions. Also, we give some applications.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$. Let $B_r = \{z \in \mathbb{C}^n : \|z\| < r\}$ and let $B = B_1$. In the case of one variable, B_r is denoted by U_r and U_1 by U. The topological closure of a subset A of $\mathbb{C}^n$ is denoted by $\overline{A}$. If $\Omega \subset \mathbb{C}^n$ is an open set, let $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. Also let $L(\mathbb{C}^n, \mathbb{C}^m)$ be the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. Let I be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$.

If $f \in H(B)$, let $D^k f(z)$ be the k-th Fréchet derivative of f at $z \in B$ and let

$$D^k f(z)(w^k) = D^k f(z)(\underbrace{w, \dots, w}_{k\text{-times}}), \quad w \in \mathbb{C}^n.$$

If $f, g \in H(B)$, we say that f is subordinate to g (write $f \prec g$) if there exists a Schwarz mapping v (i.e. $v \in H(B)$, $v(0) = 0$ and $\|v(z)\| < 1$, $z \in B$) such that $f(z) = g(v(z))$ for $z \in B$. If g is biholomorphic on B, this condition is equivalent to requiring that $f(0) = 0$ and $f(B) \subseteq g(B)$.

DEFINITION 1.1. A mapping $f : B \times [0, \infty) \to \mathbb{C}^n$ is called a subordination chain if it satisfies the following conditions:

(i) $f(\cdot, t) \in H(B)$, $t \geq 0$;

(ii) $f(\cdot, s) \prec f(\cdot, t)$ for $t \geq s \geq 0$.

Moreover, if $f(z, t)$ is a subordination chain such that $f(\cdot, t)$ is biholomorphic on B for $t \geq 0$, we say that $f(z, t)$ is a Loewner chain (or a univalent subordination chain). In this case, the condition (ii) is equivalent to the fact that there is a unique biholomorphic Schwarz mapping $v = v(z,s,t)$ such that

$$f(z,s) = f(v(z,s,t),t), \quad z \in B,\ 0 \le s \le t < \infty. \tag{1.1}$$

An important role in our discussion is played by the following sets, which are n-dimensional versions of the well known Carathéodory set in one variable:

$$\mathcal{N} = \{h \in H(B) : \ h(0) = 0,\ \mathrm{Re}\ \langle h(z), z\rangle > 0,\ z \in B \setminus \{0\}\},$$

$$\mathcal{M} = \{h \in \mathcal{N} : \ Dh(0) = I\}.$$

Recently, Graham, Hamada and Kohr [5] have proved that $\mathcal{M}$ is a compact set. In fact, the authors proved the following result:

LEMMA 1.2. *For each $r \in (0,1)$, there exists a number $M = M(r) \le 4r/(1-r)^2$ such that $\|h(z)\| \le M(r)$ for $z \in \overline{B}_r$ and $h \in \mathcal{M}$.*

Next, we recall the following result due to Pfaltzgraff [11] which yields that the solutions of the Loewner differential equation, which satisfy a normality condition, give Loewner chains. We have (cf. [11, Theorem 2.1 and Lemma 2.2], [13, Theorems 2 and 3], and [5, Theorem 1.4]):

LEMMA 1.3. *Let $h : B \times [0,\infty) \to \mathbb{C}^n$ be a mapping which satisfies the following assumptions:*
(i) *$h(\cdot,t) \in \mathcal{M},\ t \ge 0$;*
(ii) *$h(z,\cdot)$ is measurable on $[0,\infty)$, $z \in B$.*
Then there exists a unique solution $v(t) = v(z,s,t)$ of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t) \ \text{a.e.}\ t \ge s,\ v(s) = z. \tag{1.2}$$

The mapping $v(z,s,t) = \mathrm{e}^{s-t}z + \cdots$ is a biholomorphic Schwarz mapping and is Lipschitz continuous in $t \in [s,\infty)$ locally uniformly with respect to $z \in B$. In addition,

$$\lim_{t\to\infty} \mathrm{e}^t v(z,s,t) = f(z,s) \tag{1.3}$$

locally uniformly on B for $s \ge 0$, $f(\cdot,s)$ is biholomorphic on B and

$$f(z,s) = f(v(z,s,t),t), \quad z \in B,\ t \ge s \ge 0. \tag{1.4}$$

Thus $f(z,t)$ is a Loewner chain. Moreover, $f(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B$ and

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t) \ \text{a.e.}\ t \ge 0,\ \forall\ z \in B. \tag{1.5}$$

Also

$$\mathrm{e}^t \frac{\|z\|}{(1+\|z\|)^2} \le \|f(z,t)\| \le \mathrm{e}^t \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B,\ t \ge 0. \tag{1.6}$$

Note that the solution $v(z, s, t)$ of (1.2) satisfies the following semigroup property (see [11], [6]):

$$v(z, s, u) = v(v(z, s, t), t, u), \quad z \in B, \ 0 \le s \le t \le u < \infty. \tag{1.7}$$

We next provide a method that can be used to obtain univalence criteria and quasiconformal extension. The method is similar to that in [8], but is rather more general than in [8, Corollary 2.4] since it can be applied to nonnormalized subordination chains. In the case of one variable, this method is due to Becker (see [1, 2]).

2. MAIN RESULTS

We begin this section with the following useful result, which is a generalization to higher dimensions of [2, Lemma 1] (compare with [11, Theorem 2.1]):

LEMMA 2.1. *Let $h : B \times [0, \infty) \to \mathbb{C}^n$ satisfy the following conditions:*
(i) *$h(\cdot, t) \in \mathcal{N}$, $Dh(0, t) = c(t)I$ where $c : [0, \infty) \to \mathbb{C}$ is a bounded integrable function on each closed interval $[0, T]$, $T > 0$, such that*

$$\int_0^\infty \mathrm{Re}\ c(t)\mathrm{d}t = \infty; \tag{2.1}$$

(ii) *$h(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in B$.*
Then for each $s \ge 0$ and $z \in B$, the initial value problem

$$\frac{\partial v}{\partial t} = -h(v, t) \ \text{a.e. } t \ge s, \ v(z, s, s) = z, \tag{2.2}$$

has a unique solution $v = v(z, s, t)$ such that for fixed s and t, $v(\cdot, s, t)$ is a biholomorphic Schwarz mapping, $Dv(0, s, t) = (a(s)/a(t))I$ where $a(t) = \exp \int_0^t c(\tau)\mathrm{d}\tau$. Moreover, $v(z, s, \cdot)$ is a locally Lipschitz continuous function on $[s, \infty)$ locally uniformly with respect to $z \in B$. In addition, for each $s \ge 0$,

$$\lim_{t\to\infty} a(t)v(z, s, t) = f(z, s) \tag{2.3}$$

locally uniformly on B, $f(\cdot, s)$ is biholomorphic on B and

$$f(z, s) = f(v(z, s, t), t), \quad z \in B, \ t \ge s \ge 0. \tag{2.4}$$

Thus $f(z, t)$ is a Loewner chain, which is locally absolutely continuous in $t \in [0, \infty)$ locally uniformly with respect to $z \in B$ and

$$\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t) \ \text{a.e. } t \ge 0, \ \forall\ z \in B. \tag{2.5}$$

The Loewner chain given by (2.3) may be called the canonical solution of the Loewner differential equation (2.5).

Proof. First, we mention that Re $c(t) > 0$ for $t \geq 0$, according to [14, Lemma 3] and the fact that $h(\cdot, t) \in \mathcal{N}$. Let

$$\alpha(t) = \int_0^t \text{Re } c(\tau)\mathrm{d}\tau \quad \text{and} \quad \beta(t) = \int_0^t \text{Im } c(\tau)\mathrm{d}\tau, \quad t \geq 0.$$

Since c is a locally integrable function on $[0, \infty)$, it follows that α and β are locally absolutely continuous functions on $[0, \infty)$ and

$$\alpha'(t) = \text{Re } c(t) \text{ and } \beta'(t) = \text{Im } c(t) \text{ a.e. } t \geq 0.$$

Moreover, since $a(t) = \exp \int_0^t c(\tau)\mathrm{d}\tau$ and Re $c(t) > 0$ for $t \geq 0$, we deduce that $|a(\cdot)|$ is a strictly increasing function on $[0, \infty)$, $a(0) = 1$ and $|a(t)| \to \infty$ as $t \to \infty$, in view of (2.1).

Let

$$z^* = \mathrm{e}^{\mathrm{i}\beta(t)}z \text{ and } t^* = \alpha(t), \quad z \in B,\ t \geq 0.$$

Then $\|z^*\| = \|z\| < 1$ and since $\alpha(t) > 0$ for $t > 0$, it follows that t^* is a function of $[0, \infty)$ into $[0, \infty)$. Since α is strictly increasing on $[0, \infty)$, $\alpha(0) = 0$ and $\alpha(t) \to \infty$ as $t \to \infty$, we deduce that $\alpha : [0, \infty) \to [0, \infty)$ is one-to-one.

Further, let $h^* : B \times [0, \infty) \to \mathbb{C}^n$ be given by

$$h^*(z, t^*) = \frac{1}{\text{Re } c(t)}\Big[\mathrm{e}^{\mathrm{i}\beta(t)}h(\mathrm{e}^{-\mathrm{i}\beta(t)}z, t) - \mathrm{i}\text{Im } c(t)z\Big].$$

Then $h^*(\cdot, t^*) \in H(B)$, $h^*(0, t^*) = 0$ and $Dh^*(0, t^*) = I$ for $t^* \geq 0$. Since $h(z, \cdot)$ is measurable on $[0, \infty)$, it is clear that $h^*(z, \cdot)$ is also measurable on $[0, \infty)$. Moreover,

$$\text{Re } \langle h^*(z, t^*), z\rangle = \frac{1}{\text{Re } c(t)}\text{Re } \langle h(\mathrm{e}^{-\mathrm{i}\beta(t)}z, t), \mathrm{e}^{-\mathrm{i}\beta(t)}z\rangle > 0$$

for $z \in B \setminus \{0\}$ and $t \geq 0$. Consequently, taking into account Lemma 1.3, we deduce for each $s^* \geq 0$ and $z^* \in B$ that the initial value problem

$$\frac{\partial v^*}{\partial t^*} = -h^*(v^*, t^*) \text{ a.e. } t^* \geq s^*,\ v^*(z^*, s^*, t^*) = z^*, \tag{2.6}$$

has a unique solution $v^* = v^*(z^*, z^*, t^*) = \mathrm{e}^{s^*-t^*}z^* + \ldots$ such that for fixed s^* and t^*, $v^*(\cdot, s^*, t^*)$ is a biholomorphic Schwarz mapping.

Now, let

$$v(z, s, t) = \mathrm{e}^{-\mathrm{i}\beta(t)}v^*(\mathrm{e}^{\mathrm{i}\beta(s)}z, \alpha(s), \alpha(t)), \quad z \in B,\ 0 \leq s \leq t < \infty.$$

Then $v(\cdot, s, t)$ is a biholomorphic Schwarz mapping and

$$Dv(0, s, t) = \mathrm{e}^{\mathrm{i}(\beta(s)-\beta(t))}Dv^*(0, \alpha(s), \alpha(t)) = \frac{a(s)}{a(t)}I.$$

Since $v^*(z^*, s^*, t^*)$ is Lipschitz continuous with respect to $t^* \geq s^*$, the function c is bounded on each interval $[0, T]$, $T > 0$, and the functions α and β are locally absolutely continuous on $[0, \infty)$, it is easily seen that the $v(z, s, t)$ is locally Lipschitz continuous with respect to $t \geq s$. Also a simple computation and the relation (2.2) yield that

$$\frac{\partial v}{\partial t} = -h(v,t) \text{ a.e. } t \geq s,\ v(z,s,s) = z.$$

Consequently, $v = v(z,s,t)$ is a solution of the initial value problem (2.1). On the other hand, the uniqueness of solutions to the initial value problem (2.6) implies the uniqueness of solutions to (2.1). Indeed, if $u = u(z,s,t)$ is another solution of (2.1), then $u^*(z,s^*,t^*) = \mathrm{e}^{\mathrm{i}\beta(t)}u(\mathrm{e}^{-\mathrm{i}\beta(s)}z,s,t)$ satisfies the initial value problem (2.6), and thus u^* must be equal to v^*. This implies that $u \equiv v$.

Moreover, since for $s^* \geq 0$,

$$\lim_{t^*\to\infty} \mathrm{e}^{t^*}v^*(z^*,s^*,t^*) = f^*(z^*,s^*),$$

and the above limit holds locally uniformly on B and provides a Loewner chain, we deduce that $f(z,t) = f^*(\mathrm{e}^{\mathrm{i}\beta(t)}z,\alpha(t))$ is also a Loewner chain. Indeed, since $f^*(\cdot,t^*)$ is biholomorphic, it follows that $f(\cdot,t)$ is also biholomorphic on B, and since $f^*(z^*,s^*) = f^*(v^*(z^*,s^*,t^*),t^*)$ it is easy to deduce that $f(z,s) = f(v(z,s,t),t)$ for $z \in B$ and $0 \leq s \leq t < \infty$. This chain satisfies the following conditions: $f(0,t) = 0$, $Df(0,t) = a(t)I$, and for each $s \geq 0$,

$$\lim_{t\to\infty} a(t)v(z,s,t) = \lim_{t\to\infty} \mathrm{e}^{\alpha(t)}v^*(\mathrm{e}^{\mathrm{i}\beta(s)}z,\alpha(s),\alpha(t)) = f^*(\mathrm{e}^{\mathrm{i}\beta(s)}z,\alpha(s)) = f(z,s)$$

locally uniformly on B, i.e. (2.3) holds. Clearly, $f(0,s) = 0$ and $Df(0,s) = a(s)I$ for $s \geq 0$. According to Lemma 1.3, $f^*(z^*,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z^* \in B$. Moreover,

$$\frac{\partial f^*}{\partial t^*}(z^*,t^*) = Df^*(z^*,t^*)h^*(z^*,t^*) \text{ a.e. } t^* \geq 0,\ \forall\ z^* \in B. \tag{2.7}$$

Taking into account the above arguments and the fact that α is locally absolutely continuous on $[0,\infty)$, we deduce that $f(z,\cdot)$ is also locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B$. According to (2.7), we obtain for almost all $t \geq 0$ that

$$\begin{aligned}\frac{\partial f}{\partial t}(z,t) &= \mathrm{e}^{\mathrm{i}\beta(t)}\mathrm{i}\beta'(t)Df^*(\mathrm{e}^{\mathrm{i}\beta(t)}z,t^*)(z) + \frac{\partial f^*}{\partial t^*}(\mathrm{e}^{\mathrm{i}\beta(t)}z,t^*)\alpha'(t)\\ &= Df(z,t)h(z,t), \quad z \in B.\end{aligned}$$

Hence (2.5) holds, as desired. This completes the proof. □

The following result is a direct consequence of Lemma 2.1 and [11, Lemma 2.2] (compare with [4]).

Lemma 2.2. *Let* $v = v(z,s,t)$ *be the solution of* (2.2). *Then*

$$|a(t)|\frac{\|v(z,s,t)\|}{(1-\|v(z,s,t)\|)^2} \leq |a(s)|\frac{\|z\|}{(1-\|z\|)^2}$$

and

$$|a(t)|\frac{\|v(z,s,t)\|}{(1+\|v(z,s,t)\|)^2} \geq |a(s)|\frac{\|z\|}{(1+\|z\|)^2},$$

for all $z \in B$ *and* $t \geq s \geq 0$.

Proof. Let $v^*(z,\alpha(s),\alpha(t)) = \mathrm{e}^{\mathrm{i}\beta(t)}v(\mathrm{e}^{-\mathrm{i}\beta(s)}z,s,t)$ for $z \in B$ and $t \geq s \geq 0$. As in the proof of Lemma 2.1, we deduce that $v^*(z,\alpha(s),\alpha(t))$ is the solution of (2.6), and thus

$$\mathrm{e}^{\alpha(t)}\frac{\|v^*(z,\alpha(s),\alpha(t))\|}{(1-\|v^*(z,\alpha(s),\alpha(t))\|)^2} \leq \mathrm{e}^{\alpha(s)}\frac{\|z\|}{(1-\|z\|)^2},$$

and

$$\mathrm{e}^{\alpha(t)}\frac{\|v^*(z,\alpha(s),\alpha(t))\|}{(1+\|v^*(z,\alpha(s),\alpha(t))\|)^2} \geq \mathrm{e}^{\alpha(s)}\frac{\|z\|}{(1+\|z\|)^2},$$

by [11, Lemma 2.2]. The conclusion now easily follows. □

Taking into account Lemmas 2.1 and 2.2, we obtain (compare with [4]):

Lemma 2.3. *Let* $f(z,t)$ *be the Loewner chain given by* (2.3). *Then*

$$|a(t)|\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z,t)\| \leq |a(t)|\frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B,\ t \geq 0. \tag{2.8}$$

Proof. It suffices to apply the relation (2.3) and Lemma 2.2. □

We are now able to prove the main result of this paper. This result has been recently obtained in [8], in the case $a(t) = \mathrm{e}^t$, $t \geq 0$. In one variable, this result is due to Becker [2] (see also [3]). His assumptions hold on the punctured disc $U \setminus \{0\}$. But in higher dimensions, holomorphic functions have no isolated singularities, and thus we assume that our conditions hold on B. We mention that in [8, Theorem 2.1] we allowed the radius $r(t)$ of the ball $B_{r(t)}$ on which the solution was initially defined in z to vary with t; for our purpose, we shall work with $r(t) \equiv 1$. We have

Theorem 2.4. *Let* $h(z,t)$ *satisfy the conditions* (i) *and* (ii) *in* Lemma 2.1. *Also let* $f(z,t)$ *be given by* (2.3) *and* $g = g(z,t) : B\times[0,\infty) \to \mathbb{C}^n$ *be a mapping such that* $g(\cdot,t) \in H(B)$, $t \geq 0$, *and* $g(z,\cdot)$ *is locally absolutely continuous on* $[0,\infty)$ *locally uniformly with respect to* $z \in B$. *Assume* $g(z,t)$ *satisfies the differential equation*

$$\frac{\partial g}{\partial t}(z,t) = Dg(z,t)h(z,t) \ \text{a.e. } t \geq 0,\ \forall\, z \in B. \tag{2.9}$$

Then $g(z,t)$ *is a subordination chain and there is a mapping* $\Phi \in H(\mathbb{C}^n)$ *such that*

$$g(z,t) = \Phi(f(z,t)), \quad z \in B,\ t \geq 0. \tag{2.10}$$

Moreover, $g(z,t)$ *is a Loewner chain if and only if* Φ *is biholomorphic on* $\mathbb{C}^n$. *Conversely, if* $g(z,t) = \Phi(f(z,t))$, $z \in B$, $t \geq 0$, *where* Φ *is a biholomorphic* *mapping on* $\mathbb{C}^n$*, then* $g(z,t)$ *is a Loewner chain which satisfies the Loewner differential equation* (2.9)*.*

Proof. As in the proof of [11, Theorem 2.2] and [8, Lemma 2.2], we deduce that

$$g(v(z,s,t),t) = g(z,s), \quad z \in B,\ t \geq s \geq 0, \tag{2.11}$$

where $v = v(z,s,t)$ is the unique solution of the initial value problem (2.2). Indeed, let $v_{s,t}(z) = v(z,s,t)$ for $z \in B$ and $t \geq s \geq 0$, and let $g_t(z) = g(z,t)$, $z \in B$, $t \geq 0$. Fix $T > 0$. Since $g(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B$, and $g(\cdot,t) \in H(B)$ for $t \geq 0$, it follows that the mapping g is continuous on $B \times [0,\infty)$. Thus for each $\rho \in (0,1)$ there is some $K = K(\rho,T) > 0$ such that

$$\|g(z,t)\| \leq K, \quad z \in \overline{B}_\rho,\ t \in [0,T].$$

Moreover, according to the Cauchy integral formula for vector valued holomorphic functions, it is easy to deduce for each $\rho \in (0,1)$ that there is some $K' = K'(\rho,T) > 0$ such that

$$\|Dg(z,t)\| \leq K', \quad z \in \overline{B}_\rho,\ t \in [0,T]. \tag{2.12}$$

Consequently, for each $\rho \in (0,1)$ we obtain

$$\|g(z,t) - g(w,t)\| \leq K'\|z - w\|, \quad z,w \in \overline{B}_\rho,\ t \in [0,T]. \tag{2.13}$$

Further, let $h^* = h^*(z,t^*) : B \times [0,\infty) \to \mathbb{C}^n$ be given by

$$h^*(z,t^*) = \frac{1}{\mathrm{Re}\ c(t)}\left[\mathrm{e}^{\mathrm{i}\beta(t)}h(\mathrm{e}^{-\mathrm{i}\beta(t)}z,t) - \mathrm{iIm}\ c(t)z\right], \quad z \in B,\ t^* \geq 0,$$

where $t^* = \alpha(t)$ and $\beta(t)$ are given in the proof of Lemma 2.1. We have shown that $h^*(\cdot,t^*) \in \mathcal{M}$, and thus in view of Lemma 1.2, for each $\rho \in (0,1)$ there is some $M = M(\rho) > 0$ such that $\|h^*(z,t^*)\| \leq M(\rho)$ for $\|z\| \leq \rho$ and $t^* \geq 0$. Then

$$\|h(z,t)\| \leq M(\rho)\mathrm{Re}\ c(t) + \rho|\mathrm{Im}\ c(t)|, \quad z \in \overline{B}_\rho,\ t \geq 0.$$

On the other hand, taking into account the relations (2.9) and (2.12), we deduce that

$$\left\|\frac{\partial g}{\partial t}(z,t)\right\| \leq K'[M(\rho)\mathrm{Re}\ c(t) + \rho|\mathrm{Im}\ c(t)|], \quad \text{a.e. } t \in [0,T],\ \forall\ z \in \overline{B}_\rho.$$

Since $g(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$, we obtain in view of the above relation that

$$\|g(z,t_1) - g(z,t_2)\| \leq K' \int_{t_1}^{t_2} [M(\rho)\mathrm{Re}\ c(\tau) + \rho|\mathrm{Im}\ c(\tau)|]\mathrm{d}\tau,$$

and since c is bounded on $[0,T]$, there exists some constant $K^* = K^*(\rho,T) > 0$ such that

$$\|g(z,t_1) - g(z,t_2)\| \leq K^*|t_1 - t_2|, \quad z \in \overline{B}_\rho,\ t_1,t_2 \in [0,T]. \tag{2.14}$$

Next, fix $\rho \in (0,1)$ and $s \in [0,T]$. Let $q(z,t) = g(v_{s,t}(z),t)$ for $z \in \overline{B}_\rho$ and $t \in [s,T]$. According to the relations (2.13) and (2.14), and the local Lipschitz continuity of $v(z,s,\cdot)$ on $[s,\infty)$, we obtain

$$\begin{aligned} &\|q(z,t_1) - q(z,t_2)\| \\ &\le \|g(v_{s,t_1}(z),t_1) - g(v_{s,t_1}(z),t_2)\| + \|g(v_{s,t_1}(z),t_2) - g(v_{s,t_2}(z),t_2)\| \\ &\le K^*|t_1 - t_2| + K'\|v_{s,t_1}(z) - v_{s,t_2}(z)\| \\ &\le [K^* + K'L(r,T)]|t_1 - t_2| \le K^{**}|t_1 - t_2|, \quad z \in \overline{B}_\rho,\ t_1,t_2 \in [s,T]. \end{aligned}$$

Hence for each $z \in \overline{B}_\rho$, $q(z,t)$ is Lipschitz continuous with respect to $t \in [s,T]$ and in view of the relations (2.2) and (2.9) we obtain

$$\begin{aligned} &\frac{\partial q}{\partial t}(z,t) = Dg(v_{s,t}(z),t)\frac{\partial v_{s,t}}{\partial t}(z) + \frac{\partial g}{\partial t}(v_{s,t}(z),t) \\ &= -Dg(v_{s,t}(z),t)h(v_{s,t}(z),t) + Dg(v_{s,t}(z),t)h(v_{s,t}(z,t),t) = 0, \end{aligned}$$

for almost all $t \in [s,T]$. Since $q(z,\cdot)$ is absolutely continuous on $[s,T]$, we conclude that $q(z,\cdot)$ is constant on $[s,T]$, and thus $q(z,s) = q(z,t)$ i.e.

$$g(v(z,s,t),t) = g(z,s), \quad t \in [s,T],\ z \in \overline{B}_\rho.$$

Since T is arbitrary, we obtain

$$g(v(z,s,t),t) = g(z,s), \quad t \in [s,\infty),\ z \in \overline{B}_\rho.$$

Further, taking into account the identity theorem for holomorphic mappings and the above equality, we deduce (2.11), as claimed. This relation can be written equivalently, as follows

$$g_s(z) = g_t((a(t))^{-1}a(t)v_{s,t}(z)), \quad z \in B,\ t \ge s \ge 0. \tag{2.15}$$

Let $\psi_{s,t}(z) = a(t)v_{s,t}(z)$ for $z \in B$ and $0 \le s \le t < \infty$. In view of (2.3), we deduce for each $s \ge 0$ that

$$\lim_{t\to\infty} \psi_{s,t}(z) = f(z,s) \tag{2.16}$$

and the above limit holds locally uniformly on B. Moreover, $f_t(z) = f(z,t)$ satisfies the following growth result

$$|a(t)|\frac{\|z\|}{(1+\|z\|)^2} \le \|f_t(z)\| \le |a(t)|\frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B,\ t \ge 0,$$

by (2.8). Since $|a(t)|$ is strictly increasing to ∞ as $t \to \infty$, we deduce for each $m = 1,2,\ldots$ that there is some $s_m \ge 0$ such that $f_{s_m}(B) \supseteq \overline{B}_m$. Consequently, taking into account (2.16) and the above relation, we obtain for each $m = 1,2,\ldots$ that there is some $t_m \ge s_m$ such that $\psi_{s_m,t}(B) \supseteq \overline{B}_m, \quad t \ge t_m$.

Since $\{\psi_{s_m,t}\}_{t\ge t_m}$ is a family of biholomorphic mappings on B such that $\psi_{s_m,t}(0) = 0$, $D\psi_{s_m,t}(0) = a(s_m)I$ for $t \ge t_m$, and

$$|a(s_m)|\frac{\|z\|}{(1+\|z\|)^2} \le \|\psi_{s_m,t}(z)\| \le |a(s_m)|\frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B,\ t \ge t_m,$$

by Lemma 2.2, we obtain in view of [10, Theorem 2.1] that

$$\lim_{t\to\infty} \psi_{s_m,t}^{-1}(w) = f_{s_m}^{-1}(w)$$

locally uniformly on B_m for $m = 1, 2, \ldots$. According to (2.15), we have

$$g_t((a(t))^{-1}w) = g_{s_m}(\psi_{s_m,t}^{-1}(w)), \quad w \in \overline{B}_m,\ t \geq t_m.$$

Since $g_{s_m} \in H(B)$ and $\psi_{s_m,t}^{-1}(w) \to f_{s_m}^{-1}(w)$ as $t \to \infty$ locally uniformly on B_m, we deduce from the above equality that $g_t((a(t))^{-1}w) \to \Phi_m$ locally uniformly on B_m as $t \to \infty$, where Φ_m is a holomorphic mapping by Weierstrass theorem, for $m = 1, 2, \ldots$. Since m is arbitrary, we deduce in view of the identity theorem for holomorphic mappings that there is a mapping $\Phi \in H(\mathbb{C}^n)$ such that Φ is the holomorphic extension to $\mathbb{C}^n$ of Φ_m for $m = 1, 2, \ldots$ and $g_t((a(t))^{-1}w) \to \Phi$ locally uniformly on $\mathbb{C}^n$. Letting $t \to \infty$ in (2.15) and using (2.16), we conclude that

$$g_s(z) = \Phi(f_s(z)), \quad z \in B,\ s \geq 0,$$

and thus (2.10) holds, as desired.

Finally, we note that the estimate (2.8) and the fact that $|a(t)| \to \infty$ as $t \to \infty$ imply that $\bigcup_{s\geq 0} f_s(B) = \mathbb{C}^n$. This easily implies that $g(z,t)$ is a Loewner chain if and only if Φ is biholomorphic on $\mathbb{C}^n$.

Conversely, if $g(z,t) = \Phi(f(z,t))$, $z \in B$, $t \geq 0$, where Φ is a biholomorphic mapping on $\mathbb{C}^n$, then it is clear that $g(\cdot,t)$ is biholomorphic on B, $g(z,s) = g(v(z,s,t),t)$ for $z \in B$ and $t \geq 0$, and thus $g(z,t)$ is a Loewner chain. Also, it is easily seen that $g(z,t)$ satisfies the Loewner differential equation (2.9). This completes the proof. □

We next give the following consequence of Theorem 2.4, in terms of the coefficients of $g(z,t)$. In the case of one variable, this result was obtained by Becker [2]. We have

Corollary 2.5. *Let $g(z,t)$ and $h(z,t)$ satisfy the assumptions of* Theorem 2.4. *Also let $f(z,t)$ be given by* (2.3) *and let*

$$c_k(t) = \frac{1}{k!} D^k g_t(0), \quad t \geq 0,\ k \geq 0.$$

Assume that

$$\liminf_{t\to\infty} |a(t)|^{-k} \|c_k(t)\| = 0, \quad k = 2, 3, \ldots. \tag{2.17}$$

Then $g(z,t) = c_0(0) + c_1(0)(f(z,t))$, $z \in B$, $t \geq 0$.

Proof. According to Theorem 2.4, there is an entire mapping Φ such that

$$g(z,t) = \Phi(f(z,t)), \quad z \in B,\ t \geq 0. \tag{2.18}$$

Since $\Phi \in H(\mathbb{C}^n)$, it can be expanded in a power series

$$\Phi(w) = A_0 + A_1(w) + \cdots + A_k(w^k) + \ldots, \quad w \in \mathbb{C}^n,$$

where

$$A_k = \frac{1}{k!}D^k\Phi(0), \quad k \geq 0.$$

We prove that $A_k \equiv 0$ for $k \geq 2$. To this end, let

$$a_k(t) = \frac{1}{k!}D^k f_t(0), \quad t \geq 0,\ k \geq 0.$$

Then $a_0(t) = 0$ and $a_1(t) = a(t)I$, $t \geq 0$, and thus $a_1(0) = I$. Moreover, (2.18) yields

$$\sum_{k=0}^{\infty} c_k(t)(z^k) = \sum_{k=0}^{\infty} A_k(w^k), \quad z \in B,$$

where $w = \sum\limits_{k=0}^{\infty} a_k(t)(z^k)$, and hence

$$c_0(t) = A_0, \quad c_1(t) = a(t)A_1, \quad t \geq 0,$$

and

$$c_k(t) = A_1 \circ a_k(t) + [a(t)]^k A_k, \quad k \geq 2,\ t \geq 0. \tag{2.19}$$

On the other hand, taking into account Lemma 2.3 and the Cauchy integral formulas for vector valued holomorphic functions, it is easy to deduce for each $k \geq 2$ that there is some $C_k > 0$ such that

$$\|a_k(t)\| \leq C_k|a(t)|, \quad t \geq 0.$$

Indeed, since

$$a_k(t)(w^k) = \frac{1}{2\pi i}\int\limits_{|\zeta|=r} \frac{f(\zeta w, t)}{\zeta^{k+1}}\mathrm{d}\zeta, \quad \|w\| = 1,$$

for any $r \in (0,1)$, and since

$$\|f(z,t)\| \leq |a(t)|\frac{r}{(1-r)^2}, \quad \|z\| = r,\ t \geq 0,$$

we obtain the following relation

$$\|a_k(t)\| \leq |a(t)| \min_{r\in(0,1)} \frac{1}{r^{k-1}(1-r)^2} \leq |a(t)| \left[\frac{\mathrm{e}(k+1)}{2}\right]^2.$$

Letting $C_k = \mathrm{e}^2(k+1)^2/4$, the claimed conclusion follows.

Next, fix $k \geq 2$. Multiplying both sides of (2.19) by $[a(t)]^{-k}$ and using the above relation, we obtain

$$\|[a(t)]^{-k}c_k(t) - A_k\| \leq \|A_1\| \cdot |a(t)|^{-k}\|a_k(t)\| \leq C_k|a(t)|^{1-k}\|A_1\|,\ t \geq 0.$$

Further, according to (2.17), there is a sequence $\{t_m\}_{m\in\mathbb{N}}$ such that $t_m > 0$, $t_m \to \infty$ as $m \to \infty$, and

$$|a(t_m)|^{-k}\|c_k(t_m)\| \to 0 \text{ as } m \to \infty. \tag{2.20}$$

Finally, letting $m \to \infty$ in the relation

$$\|[a(t_m)]^{-k} c_k(t_m) - A_k\| \le C_k |a(t_m)|^{1-k} \|A_1\|,$$

and using (2.20) and the fact that $|a(t_m)| \to \infty$ as $m \to \infty$, we conclude that $A_k \equiv 0$. Hence

$$g(z,t) = A_0 + A_1(f(z,t)) = c_0(0) + c_1(0)(f(z,t)), \quad t \ge 0,\ z \in B,$$

as claimed. This completes the proof. □

Remark 2.6. The condition (2.17) is satisfied if $\{g(z,t)/a(t)\}_{t \ge 0}$ is a normal family on B. In particular, if $g(z,t) = a(t)z + \cdots$ satisfies the assumptions of Theorem 2.4 and $\{g(z,t)/a(t)\}_{t \ge 0}$ is a normal family on B, then $g(z,t) = f(z,t)$ for $z \in B$ and $t \ge 0$, where $f(z,t)$ is given by (2.3). Thus, we obtain (see [8, Corollary 2.3] in the case $a(t) = \mathrm{e}^t$, $t \ge 0$).

Corollary 2.7. *Let $h(z,t) = c(t)z + \cdots$ satisfy the conditions* (i) *and* (ii) *in* Lemma 2.1*, where $c : [0,\infty) \to \mathbb{C}$ is a bounded integrable function on each closed interval $[0,T]$, $T > 0$, that satisfies the relation* (2.1)*. Also let $a(t) = \exp \int_0^t c(\tau)\mathrm{d}\tau$ for $t \ge 0$, and $g : B \times [0,\infty) \to \mathbb{C}^n$ be a mapping such that $g(\cdot,t) \in H(B)$, $g(0,t) = 0$, $Dg(0,t) = a(t)I$, $t \ge 0$, and $g(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B$. Assume that $g(z,t)$ satisfies the differential equation* (2.9) *and $\{g(z,t)/a(t)\}_{t \ge 0}$ is a normal family on B. Then $g(z,t)$ is a Loewner chain.*

Finally we mention the following generalization of [8, Theorem 2.1] to non-normalized subordination chains $g(z,t) = a(t)z + \cdots$ In the case of one variable, this result is due to Becker [2, 3]. We show that with appropriate conditions on $c(t)$ and the radius of convergence $r(t)$, we can obtain subordination chains as solutions of the Loewner differential equation

$$\frac{\partial g}{\partial t}(z,t) = Dg(z,t)h(z,t), \quad \text{a.e. } t \ge 0, \forall\, z \in B_{r(t)}.$$

The proof of this result follows arguments similar to those in the proofs of Theorem 2.4 and [8, Theorem 2.1]. We shall leave it to the reader.

Theorem 2.8. *Let $h(z,t) = c(t)z + \cdots$ satisfy the conditions* (i) *and* (ii) *in* Lemma 2.1*, where $c : [0,\infty) \to \mathbb{C}$ is a bounded integrable function on each closed interval $[0,T]$, $T > 0$, that satisfies the relation* (2.1)*. Also let $a(t) = \exp \int_0^t c(\tau)\mathrm{d}\tau$ for $t \ge 0$, $f(z,t)$ be given by* (2.3) *and let $g(z,t)$ be a mapping such that for each $t \ge 0$, $g(\cdot,t) \in H(B_{r(t)})$ where $r : [0,\infty) \to (0,1]$ is a continuous function with $\limsup_{t\to\infty} r(t)|a(t)| = \infty$. Assume there exist two positive functions ρ and δ on $[0,\infty)$ such that $\rho(t) < 1$, $t \ge 0$, and for each $t_0 \ge 0$ the following conditions hold:*

(a) *$r(t) \ge \rho(t_0)$ for $t \in E_{\delta(t_0)} = [t_0 - \delta(t_0), t_0 + \delta(t_0)] \cap [0,\infty)$;*

(b) *$g(z,\cdot)$ is absolutely continuous on $E_{\delta(t_0)}$ for $z \in B_{\rho(t_0)}$, and*

$$\frac{\partial g}{\partial t}(z,t) = Dg(z,t)h(z,t), \quad \text{a.e. } t \in E_{\delta(t_0)}, \forall\, z \in B_{\rho(t_0)}.$$

Then $g(z,t)$ extends to a subordination chain, again denoted by $g(z,t)$, and there exists a mapping $\Psi : \mathbb{C}^n \to \mathbb{C}^n$ which is holomorphic such that

$$g(z,t) = \Psi(f(z,t)), \quad z \in B,\, t \geq 0.$$

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