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CONDITIONS FOR STARLIKENESS AND FOR CONVEXITY

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Abstract. In this paper we define new classes $P_{n,m}[\alpha, M]$ and we give a sufficient condition for starlikeness and also a sufficient condition for convexity of analytic functions in the unit disc.

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1. INTRODUCTION

Let $\mathcal{A}_n$, $n \in \mathbb{N}^*$, denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

which are analytic in the unit disc $\mathbf{U} = \{z \ : \ z \in \mathbb{C} \ , \ |z| < 1\}$.

For $f \in \mathcal{A}_n$ we define the differential operator $\mathbf{D}^n$ (Sălăgean [3])

$$\mathbf{D}^0 f(z) = f(z)$$
$$\mathbf{D}^1 f(z) = \mathbf{D}f(z) = zf'(z)$$
$$\mathbf{D}^{n+1} f(z) = \mathbf{D}(\mathbf{D}^n f(z)); \ n \in \mathbb{N}^* \cup \{0\}\,.$$

For $f \in \mathcal{A}_n$ we define the class $P_{n,m}[\alpha, M]$ where $\alpha \geq 0$; $M > 0$ and $m \in \mathbb{N} = \{0, 1, 2, ...\}$ by

$$P_{n,m}[\alpha, M] = \left\{ f \in \mathcal{A}_n : \ \left| (1-\alpha)\frac{\mathbf{D}^{m+1} f(z)}{z} + \alpha \frac{\mathbf{D}^{m+2} f(z)}{z} - 1 \right| < M \right\}.$$

REMARK 1. For $m = 0$, we have $P_{n,0}[\alpha, M] = P_n[\alpha, M]$ and this class was studied by Liu Jinlin [1].

Let $f(z)$ and $g(z)$ be analytic in $\mathbf{U}$. Then we say that the function $g(z)$ is subordinate to $f(z)$ in $\mathbf{U}$ if there exists an analytic function $h(z)$ in $\mathbf{U}$ such that $|h(z)| < 1$ and $g(z) = f(h(z))$. For this relation the symbol $g(z) \prec f(z)$ is used. If $f(z)$ is univalent in $\mathbf{U}$, we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(\mathbf{U}) \subset f(\mathbf{U})$. The function $f \in A_n$ is called convex if $\mathrm{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > 0$ for every $z \in U$.

DEFINITION 1. *Let the function f be in the class $\mathcal{A}_n$. Then f is said to be m-starlike, $m \in \mathbb{N}$, if it satisfies the condition*

$$\mathrm{Re}\frac{\mathbf{D}^{m+1} f(z)}{\mathbf{D}^m f(z)} > 0, \quad z \in U.$$

Lemma 2. [2] *Let* $p(z) = a + p_n z^n + \dots$ $(n \geq 1)$ *be analytic in* $\mathbf{U}$ *and let* $h(z)$ *be convex in* $\mathbf{U}$ *with* $h(0) = a$. *If* $p(z) + \frac{1}{c} zp'(z) \prec h(z)$, *where* $c \neq 0$ *and* $\operatorname{Re} c \geq 0$, *then*

$$p(z) \prec \frac{c}{n} z^{-\frac{c}{n}} \int_0^z h(t) t^{\frac{c}{n}-1} \mathrm{d}t.$$

2. MAIN RESULTS

Theorem 3. *Let* $f(z) \in P_{n,m}[\alpha, M]$. *Then*

$$\left| \frac{\mathbf{D}^{m+1} f(z)}{z} \right| \leq 1 + \frac{M}{1+n\alpha} |z|^n$$

and

$$\operatorname{Re} \frac{\mathbf{D}^{m+1} f(z)}{z} \geq 1 - \frac{M}{1+n\alpha} |z|^n , \quad z \in U. \tag{1}$$

Proof. The condition $f(z) \in P_{n,m}[\alpha, M]$ is equivalent to the subordination

$$(1-\alpha) \frac{\mathbf{D}^{m+1} f(z)}{z} + \alpha \frac{\mathbf{D}^{m+2} f(z)}{z} \prec 1 + Mz.$$

For $p(z) = \frac{\mathbf{D}^{m+1} f(z)}{z}$ we have

$$\begin{aligned} zp'(z) &= z \frac{z(\mathbf{D}^{m+1} f(z))' - \mathbf{D}^{m+1} f(z)}{z^2} = \\ &= \frac{z\mathbf{D}^{m+2} f(z) - z\mathbf{D}^{m+1} f(z)}{z^2} = \frac{\mathbf{D}^{m+2} f(z)}{z} - \frac{\mathbf{D}^{m+1} f(z)}{z} \end{aligned}$$

and

$$\frac{\mathbf{D}^{m+2} f(z)}{z} = zp'(z) + \frac{\mathbf{D}^{m+1} f(z)}{z} = zp'(z) + p(z).$$

Then $(1-\alpha) p(z) + \alpha (p(z) + zp'(z)) = p(z) + \alpha zp'(z) \prec 1 + Mz$ and for this, by Lemma 3, we have

$$\frac{\mathbf{D}^{m+1} f(z)}{z} \prec \frac{1}{n\alpha} z^{-\frac{1}{n\alpha}} \int_0^z (1+Mt)\, t^{\frac{1}{n\alpha}-1} \mathrm{d}t = 1 + \frac{M}{1+n\alpha} z.$$

If we get

$$\frac{\mathbf{D}^{m+1} f(z)}{z} = 1 + \frac{M}{1+n\alpha} \varphi(z),$$

where $\varphi(z)$ is analytic in $\mathbf{U}$ and $|\varphi(z)| \leq |z|^n$, $z \in \mathbf{U}$, we obtain

$$\left| \frac{\mathbf{D}^{m+1} f(z)}{z} \right| \leq 1 + \frac{M}{1+n\alpha} |z|^n$$

and

$$\operatorname{Re} \frac{\mathbf{D}^{m+1} f(z)}{z} \geq 1 - \frac{M}{1+n\alpha} |z|^n .$$

Remark 2. If $m = 0$ we obtain the result of Jinlin: Let $f(z) \in P_n[\alpha, M]$; then f is starlike in $\mathbf{U}$, for $M \leq \frac{(1+n)(1+n\alpha)}{\sqrt{1+(n+1)^2}}$.

If $m = 1$, for the class

$$P_{n,1}[\alpha, M] = \left\{ f \in \mathcal{A}_n : \left| (1-\alpha)\frac{\mathbf{D}^2 f(z)}{z} + \alpha \frac{\mathbf{D}^3 f(z)}{z} - 1 \right| < M \right\}$$

we obtain the next theorem:

Theorem 4. *Let $f(z) \in P_{n,1}[\alpha, M]$. If $M \leq \frac{(1+n)(1+n\alpha)}{\sqrt{1+(n+1)^2}}$, then f is convex in U.*

Proof. Since $f(z) \in P_{n,1}[\alpha, M]$, it follows from Theorem 4 that

$$\left| \frac{\mathbf{D}^2 f(z)}{z} - 1 \right| \leq \frac{M}{1+n\alpha}$$

is equivalent to

$$\left| \arg \frac{\mathbf{D}^2 f(z)}{z} \right| < \arcsin \frac{M}{1+n\alpha} \leq \arcsin \frac{1+n}{\sqrt{1+(n+1)^2}}$$

and

$$\left| \arg \frac{\mathbf{D} f(z)}{z} \right| < \arcsin \frac{M}{(1+n)(1+n\alpha)} \leq \arcsin \frac{1}{\sqrt{1+(n+1)^2}}.$$

Using this inequality, we obtain

$$\begin{aligned} \left| \arg \frac{\mathbf{D}^2 f(z)}{\mathbf{D} f(z)} \right| &\leq \left| \arg \frac{\mathbf{D}^2 f(z)}{z} \right| + \left| \arg \frac{\mathbf{D} f(z)}{z} \right| < \\ &< \arcsin \frac{1+n}{\sqrt{1+(n+1)^2}} + \arcsin \frac{1}{\sqrt{1+(n+1)^2}} = \frac{\pi}{2}, \ z \in U, \end{aligned}$$

which implies that $\mathrm{Re}\frac{\mathbf{D}^2 f(z)}{\mathbf{D} f(z)} > 0$ and thus

$$\mathrm{Re}\frac{z\left(zf'(z)\right)}{zf'(z)} = Re\frac{z(f'(z)+zf''(z))}{zf'(z)} = Re(1+\frac{zf''(z)}{f'(z)}) > 0.$$

Remark 3. Let $f(z) \in P_{n,m}[\alpha, M]$. If $M \leq \frac{(1+n)(1+n\alpha)}{\sqrt{1+(n+1)^2}}$, then

$$\left| \arg \frac{\mathbf{D}^{m+1} f(z)}{\mathbf{D}^m f(z)} \right| < \frac{\pi}{2},$$

hence f is an n-starlike function.

Theorem 5. *Let $c > -1$ and let $f(z) \in P_{n,m}[\alpha, M]$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) \mathrm{d}t \tag{2}$$

belongs to $P_{n,m}\left[\frac{1}{c+1}, \frac{M}{1+n\alpha}\right]$. The result is sharp.

Proof. By (2) we have $F'(z)+\frac{1}{c+1}zF''(z)=f'(z)\prec 1+\frac{M}{1+n\alpha}z$, which shows that $F(z)\in P_{n,m}\left[\frac{1}{c+1},\frac{M}{1+n\alpha}\right]$.

Theorem 6. *Let $c>-1$ and $\alpha>0$. If $F(z)\in P_{n,m}[\alpha,M]$, then the function defined by* (2) *satisfies $|f'(z)-1|<M$ for $z\in U$.*

Proof. Since $F(z)\in P_{n,m}[\alpha,M]$, we have

$$F'(z)+\frac{1}{c+1}zF''(z)=f'(z)\prec 1+ Mz \tag{3}$$

and

$$F'(z)\prec 1+\frac{M}{1+n\alpha}z\prec 1+Mz.$$

From (3), we get

$$\begin{aligned} f'(z) &= \frac{1}{\alpha(c+1)}\left\{\left[F'(z)+\alpha zF''(z)\right]+\left[\alpha(c+1)-1\right]F'(z)\right\}\prec \\ &\prec \frac{1}{\alpha(c+1)}\left\{1+Mz+\left[\alpha(c+1)-1\right](1+Mz)\right\}=1+Mz, \end{aligned}$$

which implies that $|f'(z)-1|\leq M|z|<M$ for $z\in U$.

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