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OPENNESS RESULTS FOR MULTIFUNCTIONS VIA AN ABSTRACT SUBDIFFERENTIAL

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Abstract. In this paper we establish some openness-type results for set-valued maps in Banach spaces, using a coderivarive which is defined via abstract subdifferentials. Mainly, results of openness at a linear rate are obtained. Then we apply these results to derive a necessary optimality condition for a nonlinear optimization problem, without any convexity assumptions.

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1. INTRODUCTION AND PRELIMINARY RESULTS

The aim of this paper is to obtain openness results for set-valued maps using an abstract subdifferential; one of this results will be used to derive necessary optimality conditions for a nonlinear optimization problem. This kind of results have been obtained, for example, by B. Mordukhovich in [11] for the Mordukhovich subdifferential and A. D. Ioffe in [5] for approximative subdifferential. Both papers [11] and [5] concerning with the finite dimensional spaces. A very significant step on this topic is done in the paper [6] in terms of abstract subdifferential. Here we try to use some similar conditions on a coderivative associated with an abstract subdifferential in Banach spaces for obtaining openness results with a more accurate estimates for the neighborhoods involved.

The notations are basically standard. By X, Y, Z we denote linear normed spaces, by $L(X, Y)$ the space of all continuous linear operators from X into Y and by X^* the topological dual of the space X; w is the weak convergence on X and w^* is the weak-star convergence on X^*; if $x \in X$ the open ball with center x and radius $r > 0$ is denoted by $B(x, r)$ and $\mathcal{V}(x)$ is the filter of neighborhoods of x; by S_X we mean the unit sphere of X, and by U_X the closed unit ball of X. On the product space $X \times Y$ we consider the sum norm. If S is a subset of X we denote by $\overline{S}$ the closure of S; if $x \in X$, we denote the distance from x to S by $d(x, S) = \inf_{y \in S} d(x, y)$ and by d_S the distance function with respect to S, $d_S(x) = d(x, S)$ for every $x \in X$ (by convention, $d(x, \emptyset) = \infty$); I_S is the indicator function of S ($I_S(x) = 0$, if $x \in S$ and $I_S(x) = \infty$, if $x \notin S$).

For $r > 0$ we note $B(S, r) := \{x \in X \mid d(x, S) < r\}$; of course, $B(\{x\}, r) = B(x, r)$. If $f : X \to \mathbb{R} \cup \{\infty\}$ is a function we write, as usual, $\operatorname{Dom} f = \{x \in$

$X \mid f(x) < \infty\}$ for the domain of f and $\operatorname{epi} f = \{(x,\alpha) \in X \times \mathbb{R} \mid f(x) \leq \alpha\}$ for the epigraph of f.

Next we use an abstract subdifferential (see, e.g., [12]) and, for a set-valued map, the associated coderivative. In the sequel ∂ is a map which associates to any lower semicontinuous (lsc for short) function f from a Banach space X to $\mathbb{R} \cup \{\infty\}$, and to any $x \in X$ a subset of X^*, denoted by $\partial f(x)$. The map ∂ has the following basic properties:

(P1) $f(x) = \infty$ implies $\partial f(x) = \emptyset$;

(P2) If there exists $V \in \mathcal{V}(x)$ such that $f|_V = g|_V$ then $\partial f(x) = \partial g(x)$;

(P3) If the function f is convex then ∂f is the subdifferential in the sense of convex analysis;

(P4) If $x \in \operatorname{Dom} f$ is a local minimum point for f, then $0 \in \partial f(x)$;

(P5) If $X = Y \times Z$ and $f(y,z) = g(y) + h(z)$, then $\partial f(y,z) = \partial g(y) \times \partial h(z)$;

(P6) $\partial(\alpha f)(x) = \alpha \partial f(x)$, for every $\alpha > 0$.

In fact we shall use only (P3), (P4), (P5) and (P6) in the sequel, but we preferred to give a more comprehensive definition of this notion.

Remark 1. The main subdifferential types (called in [12] elementary subdifferentials): the Fréchet subdifferential, the Dini-Hadamard subdifferential, the viscosity subdifferentials have the properties (P1)–(P6).

If $S \subset X$ is a closed set and $x \in S$, we define the normal cone to $S \subset X$ at x by $N(S,x) := \partial I_S(x)$ (the function I_S is lsc, so the subdifferential is well defined). From properties (P4) and (P6), $N(S,x)$ defined above is a cone. If A is a convex set, from (P3), $N(A,x) = \{x^* \in X^* \mid x^*(y-x) \leq 0, \forall y \in A\}$.

If $A \subset X, B \subset Y$, $(x,y) \in X \times Y$, then $I_{A\times B}(x,y) = I_A(x) + I_B(y)$. If A and B are closed and $(a,b) \in A \times B$, then $N(A \times B, (a,b)) \overset{(P5)}{=} N(A,a) \times N(B,b)$.

Since we have a normal cone notion we can define the cone $T(S,x) := \{u \in X \mid x^*(u) \leq 0, \forall x^* \in N(S,x)\}$, called the tangent cone to the set S at x.

We suppose that ∂ enjoys the next sum basic principle (a space on which this principle holds is called a $\partial-$trustworthy space: see, e.g., [3] and [7]):

(P7) Let $f_1, f_{2,...}, f_n : X \to \mathbb{R} \cup \{+\infty\}$ be lsc functions. If $f_1, f, ..., f_n$ are all, but at most one, lipschitzian, and x is a local minimum point for the function $\sum\limits_{i=1}^{n} f_i$, then for every $\varepsilon > 0$, there exist $x_i \in B(x,\varepsilon)$, and $x_i^* \in \partial f_i(x_i)$ such that $|f_i(x_i) - f_i(x)| < \varepsilon$, for every $i = \overline{1,n}$ and $\left\| \sum\limits_{i=1}^{n} x_i^* \right\| < \varepsilon$.

In some specific situations we also suppose that the subdifferential ∂ has the next stability property:

(P8) $\limsup\limits_{y \to x} \partial f(y) = \partial f(x)$, for every lsc function f and for every $x \in X$, where by $\limsup\limits_{y\to x} \partial f(y)$ we mean $\{x^* \in X^* \mid \exists x_n \to x, \exists x_n^* \xrightarrow{w^*} x^*, \forall n \in \mathbb{N}, x_n^* \in \partial f(x_n)\}$.

For a multifunction $F : X \rightrightarrows Y$ we denote the domain and the graph by $\operatorname{Dom} F = \{x \in X \mid F(x) \neq \emptyset\}$ and $\operatorname{Gr} F = \{(x,y) \mid y \in F(x)\}$, respectively. $F^{-1} : Y \rightrightarrows X$ is the multifunction given by the relation $(y,x) \in \operatorname{Gr} F^{-1}$ if and only if $(x,y) \in \operatorname{Gr} F$. If $A \subset X$, $F(A) := \bigcup_{x \in A} F(x)$.

Definition 1. Let $F : X \rightrightarrows Y$ be a multifunction with closed graph and $(x,y) \in \operatorname{Gr} F$. The derivative of F at (x,y) associated with the subdifferential ∂ is the multifunction $DF(x,y) : X \rightrightarrows Y$ given by

$$\operatorname{Gr} DF(x,y) = T(\operatorname{Gr} F, (x,y)). \tag{1}$$

The coderivative of F at (x,y) associated with the subdifferential ∂ is the multifunction $D^*F(x.y) : Y^* \rightrightarrows X^*$ given by

$$D^*F(x,y)(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N(\operatorname{Gr} F, (x,y))\}. \tag{2}$$

2. MAIN RESULTS

We recall the notions we use below, using mainly the terms from [2].

Definition 2. A multifunction $F : X \rightrightarrows Y$ is said to be open at a linear rate ($a > 0$) around $(x,y) \in \operatorname{Gr} F$ if there exist $a > 0$, $r > 0$ and $\tau > 0$ such that, for each $(u,v) \in (B(x,r) \times B(y,r)) \cap \operatorname{Gr} F$ and for each $t \in (0,\tau]$, $B(v,at) \subset F(B(u,t))$.

Definition 3. A multifunction $F : X \rightrightarrows Y$ is said to be metrically regular around $(x,y) \in \operatorname{Gr} F$ if there exist $a_1 > 0$ and $r_1 > 0$ such that, for each $u \in B(x,r_1)$ and for each $v \in B(y,r_1)$ with $d(v,F(u)) \leq r_1$, we have $d(u,F^{-1}(v)) \leq a_1 d(v,F(u))$.

The following result is known (see [2]). For the convenience of the reader we present here a proof of it.

Lemma 1. *A multifunction $F : X \rightrightarrows Y$ with closed graph is open at a linear rate around $(x,y) \in \operatorname{Gr} F$ if and only if F is metrically regular around (x,y).*

Proof. We use the notations from Definitions 2 and 3. Suppose that F is open at a linear rate around (x,y). Let $\varepsilon \in (0,a)$; take $a_1 = 1/(a-\varepsilon)$, $r_1 \in (0,r/2)$ such that $r_1 a_1 \leq \tau$. Let $u \in B(x,r_1)$, $v \in B(y,r_1)$ with $d(v,F(u)) \leq r_1$ be fixed. We want to prove that $d(u,F^{-1}(v)) \leq a_1 d(v,F(u))$. If $v \in F(u)$ the inequality is obvious. Else, if $v \notin F(u)$, consider $\alpha \in (0, \min(r/2 - r_1, \varepsilon a_1 d(v,F(u))))$. There exists $v' \in F(u)$ such that

$$\left\|v' - v\right\| < d(v,F(u)) + \alpha \leq a d(v,F(u))/(a-\varepsilon).$$

Moreover,

$$\left\|v' - y\right\| \leq \left\|v' - v\right\| + \|v - y\| < r/2 - r_1 + r_1 + r_1 < r.$$

Therefore, $(u,v') \in (B(x,r_1) \times B(y,r_1)) \cap \operatorname{Gr} F$ and for $t = d(v,F(u))/(a-\varepsilon) \leq \tau$, using Definition 2, we can write

$$v \in B(v', a d(v,F(u))/(a-\varepsilon)) \subset F(B(u, d(v,F(u))/(a-\varepsilon))),$$

whence there exists $u' \in B(u, a_1 d(v, F(u)))$ with $v \in F(u')$; then $u' \in F^{-1}(v)$ and $d(u, F^{-1}(v)) \le \|u - u'\| \le a_1 d(v, F(u))$, i.e. the conclusion.

Suppose now that F is metrically regular around (x, y). We choose $\varepsilon > 0$ and take $r = r_1/2$, $a \in (0, 1/(a_1 + \varepsilon))$ and $\tau = r_1 a_1/2$. Let $(u, v) \in (B(x, r) \times B(y, r)) \cap \operatorname{Gr} F$, $t \in (0, \tau]$ and $v' \in B(v, at)$. If $v' \in F(u)$ we have nothing to prove. In the contrary case,

$$d(v', F(u)) \le \left\|v' - v\right\| < at < (1/a_1)(r_1 a_1/2) < r_1.$$

Moreover,

$$\left\|v' - y\right\| \le \left\|v' - v\right\| + \|v - y\| < r_1/2 + r_1/2 = r_1.$$

Consequently, $(u, v) \in B(x, r_1) \times B(y, r_1)$ and $d(v', F(u)) \le r_1$. Using now Definition 3, we have that

$$d(u, F^{-1}(v')) \le a_1 d(v', F(u)) < (a_1 + \varepsilon/2) d(v', F(u))$$

$(d(v', F(u)) > 0$ because $F(u)$ is closed, the graph of F being closed). We obtain that $d(u, F^{-1}(v')) < (a_1 + \varepsilon/2)at < t$, whence there exists $u' \in F^{-1}(v')$ such that $\|u - u'\| < t$, i.e. $v' \in F(B(u, t))$. □

We also use the following definition.

Definition 4. We say that a closed subset S of a Banach space X is proximinal if for every $x \in X$ there exists an element $x' \in S$ such that $d(x, S) = \|x - x'\|$.

It is well known (and easy to prove) that any closed set in a finite dimensional normed space is proximinal; also, if S is compact (weakly compact) in a linear normed space X or S is weakly closed and convex and X is reflexive, then S is proximinal.

Proposition 1. *Let $f : X \to \mathbb{R}$ be a Lipschitz function and $S \subset X$ a closed set. If $x \in S$ is a local minimum point for f on S, then for every $\varepsilon > 0$ there exist $x_1 \in B(x, \varepsilon), x_2 \in B(x, \varepsilon) \cap S$ with $|f(x_1) - f(x)| < \varepsilon$ and $x_1^* \in \partial f(x_1), x_2^* \in N(S, x_2)$ such that $\|x_1^* + x_2^*\| < \varepsilon$.*

Proof. Since x is a minimum local point for f on S, it follows that x is a minimum local point on X for $f + I_S$, so, by (P7), there exist $x_1 \in B(x, \varepsilon), x_2 \in B(x, \varepsilon)$, with $|f(x_1) - f(x)| < \varepsilon, |I_S(x_2) - I_S(x)| < \varepsilon$, and $x_1^* \in \partial f(x_1)$, $x_2^* \in \partial I_S(x_2)$ such that $\|x_1^* + x_2^*\| < \varepsilon$. Since $|I_S(x_2) - I_S(x) < \varepsilon|$, we have that $x_2 \in B(x, \varepsilon) \cap S$. The relation $x_2^* \in \partial I_S(x_2)$ and the definition of the normal cone ensure that $x_2^* \in N(S, x_2)$, i.e. the conclusion. □

Now, we are able to prove our main results.

Theorem 1. *Let X, Y be Banach spaces, $F : X \rightrightarrows Y$ be a multifunction with closed graph and $(x, y) \in \operatorname{Gr} F$. Suppose that $a > 0, r > 0$ are such that:*

a) $B(x, r) \subset \operatorname{Dom} F$, *$F$ is upper semicontinuous (usc for short) on* $B(x, r)$;

b) *for each* $x' \in B(x,r)$ *and* $y' \in B(y, 2ar) \cap F(x')$,

$$a \le \inf\{\|x^*\| \mid x^* \in D^*F(x',y')(y^*), y^* \in S_{Y^*}\};$$

c) $F(x')$ *is proximinal for every* $x' \in B(x,r)$.

Then, for every $r' \in (0,r), B(y, ar') \subset F(B(x,r'))$.

Proof. Let $r' \in (0,r)$ be fixed and let $v \in B(y,ar')$. Suppose that $v \notin F(B(x,r'))$, which means that for all $u \in B(x,r'), v \notin F(u)$; since $F(u)$ is closed ($\mathrm{Gr}\, F$ is closed), $d(v,F(u)) > 0$. We define $f : \overline{B(x,r')} \to \mathbb{R}$, $f(u) := d(v,F(u))$. We show that f is lsc on $\overline{B(x,r')} \subset \mathrm{Dom}\, F$. Let $u \in \overline{B(x,r')}$. If $f(u) = 0$ the property is obvious; consider $0 < \lambda < f(u) = d(v,F(u))$; there exists $\theta > 0$ such that $F(u) \cap \overline{B(v, \lambda + \theta)} = \emptyset$. Since F is usc at u we can find (see [9]) a neighborhood U of u s.t. for all $u' \in U$ the relation $F(u') \cap \overline{B(v, \lambda+\theta)} = \emptyset$ is true. It implies that $d(v,F(u')) \ge \lambda + \theta > \lambda$, for all $u' \in U$, whence f is lsc. Now, we can apply Ekeland's variational principle (in the form given in [1]) for the function f with $x_0 = x, \varepsilon = a'$, where $\|y - v\|/r' < a' < a$. There exists $u_a \in \overline{B(x,r')}$ which satisfies the relations:

$$d(v,F(u_a)) \le d(v,F(x)) - a' \|x - u_a\| \tag{3}$$

$$d(v,F(u_a)) < d(v,F(u)) + a' \|u - u_a\|, \forall u \in \overline{B(x,r')} \backslash \{u_a\} \tag{4}$$

If $\|u_a - x\| = r'$, by (3), we can write $a'r' \le d(v,F(x)) \le \|y - v\| < a'r'$, which is a contradiction; then $u_a \in B(x,r')$ and $u_a \in \mathrm{Dom}\, F$ (we can obtain this also from $d(v,F(u_a)) < \infty$). Hypothesis c) implies that there exists $v_a \in F(u_a)$ such that $d(v,F(u_a)) = \|v - v_a\| > 0$. We consider $h : X \times Y \to \mathbb{R}$, $h(u,y') := \|v - y'\| + a' \|u - u_a\|$. Let $(u,y') \in (B(x,r') \times Y) \cap \mathrm{Gr}\, F$; using (4), we have

$$h(u,y') \ge d(v,F(u)) + a' \|u - u_a\| \ge d(v,F(u_a)) = h(u_a,v_a),$$

whence (u_a,v_a) is a local minimum point for h on $\mathrm{Gr}\, F$; taking into account that h is Lipschitz, we can apply Proposition 1 with an ε which satisfies the inequalities $0 < \varepsilon < \min(r - r', 2ar - 2ar', \|v - v_a\|, (a - a')/(1+a))$. We obtain that there exist $(x_1,y_1) \in B((u_a,v_a),\varepsilon), (x_2,y_2) \in B((u_a,v_a),\varepsilon) \cap \mathrm{Gr}\, F$, $(x_1^*, y_1^*) \in \partial h(x_1,y_1)$, $(x_2^*,y_2^*) \in N(\mathrm{Gr}\, F, (x_2,y_2))$ such that $\max(\|x_1^* + x_2^*\|, \|y_1^* + y_2^*\|) < \varepsilon$. We have that $\|y_1 - v_a\| < \varepsilon < \|v - v_a\|$, whence $y_1 \neq v$; since $(x_1^*,y_1^*) \in \partial h(x_1,y_1)$ and h is a sum of convex and continuous functions, it follows that

$$(x_1^*,y_1^*) \in \{0_{X^*}\} \times S_{Y^*} + a'(U_{X^*} \times \{0_{Y^*}\}),$$

whence $\|y_1^*\| = 1$ and $\|x_1^*\| \le a'$. From $\|y_1^* + y_2^*\| < \varepsilon$ we have $\|y_2^*\| > \|y_1^*\| - \varepsilon > 0$ ($\varepsilon < 1$), whence $y_2^* \neq 0$. $N(\mathrm{Gr}\, F, (x_2,y_2))$ is a cone and $(x_2^*,y_2^*) \in N(\mathrm{Gr}\, F, (x_2,y_2))$; we obtain that

$$(x_2^*/\|y_2^*\|, y_2^*/\|y_2^*\|) \in N(\mathrm{Gr}\, F, (x_2,y_2)),$$

whence

$$x_2^*/\|y_2^*\| \in D^*F(x_2,y_2)(-y_2^*/\|y_2^*\|).$$

Clearly, $y_2^*/\|y_2^*\| \in S_{Y^*}$ and, moreover, $\|x_2^*\| < \varepsilon + \|x_1^*\| \le \varepsilon + a'$; taking into account the choice of ε, it implies that $\|x_2^*\|/\|y_2^*\| < (\varepsilon + a')/(1-\varepsilon) < a$. We can write

$$\|x_2 - x\| \le \|x_2 - u_a\| + \|u_a - x\| \le \varepsilon + r' < r$$

and

$$\|y_2 - y\| \le \|y_2 - v_a\| + \|v_a - y\| < \varepsilon + \|v_a - v\| + \|v - y\|$$
$$< \varepsilon + d(v, F(u_a)) + ar' \overset{(3)}{<} \varepsilon + d(v, F(x)) + ar' < \varepsilon + 2ar' < 2ar,$$

which is in contradiction with the assumption b) on the coderivative. The proof is complete. □

REMARK 2. If X, Y are finite dimensional spaces, then we can remove the assumption a) on the set-valued map F from the preceding theorem. Indeed, in the proof we used that $B(x,r) \subset \operatorname{Dom} F$ and that F usc on $B(x,r)$ in order to prove that the function f is lsc. On finite dimensional spaces, if we take $\lambda < f(u) = d(v, F(u))$, we have $F(u) \cap \overline{B(v,\lambda)} = \emptyset$, so $\operatorname{Gr} F \cap (\{u\} \times \overline{B(v,\lambda)}) = \emptyset$. Taking into account that $\operatorname{Gr} F$ is a closed set and $\{u\} \times \overline{B(v,\lambda)}$ is a compact set, there exists $\varepsilon > 0$ such that $\operatorname{Gr} F \cap (B(u,\varepsilon) \times B(v, \lambda + \varepsilon)) = \emptyset$, whence, for all $u' \in B(u,\varepsilon), F(u') \cap B(v, \lambda+\varepsilon) = \emptyset$, whence $d(v, F(u')) > \lambda$.

A similar remark can be formulated concerning Theorems 2, 3 and Corollary 2 given below.

THEOREM 2. *Let X, Y be Banach spaces, $F : X \rightrightarrows Y$ be a multifunction with closed graph and $x \in \operatorname{Dom} F$. Suppose that there exist $a > 0$ and $r > 0$ such that:*

a) *$B(x,r) \subset \operatorname{Dom} F$, F is usc on $B(x,r)$;*

b) *for each $x' \in B(x,r)$ and $y' \in B(F(x), 2ar) \cap F(x')$,*

$$a \le \inf\{\|x^*\| \mid x^* \in D^*F(x',y')(y^*), y^* \in S_{Y^*}\};$$

c) *$F(x')$ is proximinal for every $x' \in B(x,r)$.*

Then, for every $r' \in (0,r)$, $B(F(x), ar') \subset F(B(x,r'))$.

Proof. The conclusion follows from Theorem 1 applied in each point $y \in F(x)$. □

The next result gives sufficient conditions for F to be open at a specific linear rate.

THEOREM 3. *Let X, Y be Banach spaces, $F : X \rightrightarrows Y$ be a multifunction with closed graph and $(x,y) \in \operatorname{Gr} F$. Suppose that there exist $a > 0, r > 0, \gamma > 0$ such that:*

a) *$B(x,r) \subset \operatorname{Dom} F$, F is usc on $B(x,r)$;*

b) *for each $x' \in B(x,r)$ and $y' \in B(y, 2ar + \gamma) \cap F(x')$,*

$$a \le \inf\{\|x^*\| \mid x^* \in D^*F(x',y')(y^*), y^* \in S_{Y^*}\};$$

c) *$F(x')$ is proximinal for every $x' \in B(x,r)$.*

Then F is open at the linear rate a around the point (x,y).

Proof. First, we prove that $B(F(x'') \cap B(y,\gamma), ar'') \subset F(B(x'',r''))$, for all pairs (x'',r'') which satisfies $B(x'',r'') \subset B(x,r')$, where $0 < r' < r/2$. Taking a pair (x'',r'') such that $B(x'',r'') \subset B(x,r')$, we have $r'' \le r' < r$. Let $v \in B(F(x'') \cap B(y,\gamma), ar'')$; there exists $y'' \in F(x'') \cap B(y,\gamma)$ such that $\|y''-v\| < ar''$. Suppose that $v \notin \overline{F(B(x'',r''))}$. We apply now Ekeland's variational principle for $d(v,F(\cdot))$ on $\overline{B(x'',r'')}$, with $x_0 = x'', \varepsilon = a'$, where $\|y''-v\|/r'' < a' < a$. There exists $u_a \in \overline{B(x'',r'')}$, which satisfies the relations:

$$d(v,F(u_a)) \le d(v,F(x'')) - a' \left\|x''-u_a\right\|, \tag{5}$$

$$d(v,F(u_a)) < d(v,F(u)) + a' \left\|u-u_a\right\|, \forall u \in \overline{B(x'',r'')} \backslash \{u_a\}. \tag{6}$$

If $\|u_a - x''\| = r''$, from (5), we obtain that $a'r'' \le d(v,F(x)) \le \|y-v\| < a'r''$, which is absurd, whence $u_a \in B(x,r')$ and $u_a \in \operatorname{Dom} F$. From hypothesis c), there exists an element $v_a \in F(u_a)$ such that $d(v,F(u_a)) = \|v-v_a\| > 0$. Consider the function $h : X \times Y \to \mathbb{R}$, $h(u,y') := \|v-y'\| + a'\|u-u_a\|$. Let $(u,y') \in (B(x'',r'') \times Y) \cap \operatorname{Gr} F$; we have

$$h(u,y') \ge d(v,F(u)) + a'\|u-u_a\| \ge d(v,F(u_a)) = h(u_a,v_a),$$

whence the point (u_a,v_a) is a local minimum on $\operatorname{Gr} F$ for h; but h is Lipschitz and we can apply Proposition 1 for an ε which satisfies the inequalities $0 < \varepsilon < \min(r-2r', 2ar-2ar'', \|v-v_a\|, (a-a')/(1+a))$. There exist $(x_1,y_1) \in B((u_a,v_a),\varepsilon), (x_2,y_2) \in B((u_a,v_a),\varepsilon) \cap \operatorname{Gr} F$, $(x_1^*,y_1^*) \in \partial h(x_1,y_1)$, $(x_2^*,y_2^*) \in N(\operatorname{Gr}, (x_2,y_2))$ such that $\max(\|x_1^*+x_2^*\|, \|y_1^*+y_2^*\|) < \varepsilon$. We have $\|y_1-v_a\| < \varepsilon < \|v-v_a\|$, whence $y_1 \ne v$; moreover, $(x_1^*,y_1^*) \in \partial h(x_1,y_1)$, so

$$(x_1^*,y_1^*) \in \{0_{X^*}\} \times S_{Y^*} + a'(U_{X^*} \times \{0_{Y^*}\})$$

whence $\|y_1^*\| = 1$ and $\|x_1^*\| \le a'$. The relation $\|y_1^*+y_2^*\| < \varepsilon$ implies that $\|y_2^*\| > \|y_1^*\| - \varepsilon > 0$ $(\varepsilon < 1)$, so $y_2^* \ne 0$. Since $(x_2^*,y_2^*) \in N(\operatorname{Gr} F,(x_2,y_2))$, we have

$$(x_2^*/\|y_2^*\|, y_2^*/\|y_2^*\|) \in N(\operatorname{Gr} F,(x_2,y_2)),$$

which means that

$$x_2^*/\|y_2^*\| \in D^*F(x_2,y_2)(-y_2^*/\|y_2^*\|).$$

Clearly, $y_2^*/\|y_2^*\| \in S_{Y*}$. We can write $\|x_2^*\| < \varepsilon + \|x_1^*\| < \varepsilon + a'$ whence, $\|x_2^*\|/\|y_2^*\| < (\varepsilon+a')/(1-\varepsilon) < a$ (from the choice of ε); we obtain

$$\|x_2 - x\| \le \|x_2-u_a\| + \left\|u_a-x''\right\| + \left\|x''-x\right\| \le \varepsilon + r'' + r' < r$$

and

$$\begin{aligned}\|y_2 - y\| &\le \|y_2 - v_a\| + \|v_a - y\| \\ &< \varepsilon + \|v_a - v\| + \left\|v - y''\right\| + \left\|y'' - y\right\| \\ &< \varepsilon + d(v, F(u_a)) + ar'' + \gamma \overset{(5)}{<} \varepsilon + d(v, F(x'')) + ar'' + \gamma \\ &< \varepsilon + 2ar'' + \gamma < 2ar + \gamma,\end{aligned}$$

which is in contradiction with hypothesis b) on the coderivative. To complete the proof, we take $\rho = \min(r'/2, \gamma/2)$ and $\tau \in (0, r'/2)$; let $(u, v) \in (B(x, \rho) \times B(y, \rho)) \cap \operatorname{Gr} F$ and $t \in (0, \tau]$ be all fixed. We have $B(u, t) \subset B(x, r')$ and from the above part of the proof we can write $B(v, at) \subset F(B(u, t))$, i.e. the definition of openness at linear rate a with $r = \rho$. $\square$

Let us to compare the above results with Theorems 1 and 1a from [6, Chapter 3]. In our results the hypotheses are stronger, but we obtain a more accurate conclusion by means of the exact description of the neighborhoods involved: in this result it is indicated where the condition on the coderivative should take place and where the openness (regularity) holds. Taking into account these considerations, the results we mention are independent.

3. AN APPLICATION TO AN OPTIMIZATION PROBLEM

In this section we consider that X is an arbitrary Banach space, Y, Z are finite dimensional spaces, partially ordered by the convex closed non-empty pointed cones Y_+, respectively Z_+, with the relations $y_1 \le y_2$ if and only if $y_2 - y_1 \in Y_+$ $(y_1, y_2 \in Y)$, respectively, $z_1 \le z_2$ if and only if $z_2 - z_1 \in Z_+$ $(z_1, z_2 \in Z)$. Moreover, we consider that Y_+ has non-empty interior and we write $y_1 < y_2$ for $y_2 - y_1 \in \operatorname{int} Y_+$ $(y_1, y_2 \in Y)$.

Let $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ be multifunctions with closed graph. We define $F \times G : X \rightrightarrows Y \times Z$, $(F \times G)(x) = F(x) \times G(x)$, for all $x \in X$ (consider $A \times \emptyset = \emptyset$). Let us consider the following optimization problem (also discussed, e.g., in [4], [10]):

$$(\mathrm{P})\ \ \min F(x);\ \ 0 \in G(x) + Z_+.$$

Definition 5. A point $(x_0, y_0) \in \operatorname{Gr} F$ is called weak local minimum point for the problem (P) if $0 \in G(x_0) + Z_+$ and there exists a neighborhood U of x_0 such that, for all $x \in U$ with $0 \in G(x) + Z_+$, $(y_0 - F(x)) \cap \operatorname{int} Y_+ = \emptyset$ (consider $\emptyset \pm A = \emptyset$ for every set A).

Remark 3. a) The above definition is equivalent to the following condition: there exists a neighborhood U of x_0 such that for every point x in U with $0 \in G(x) + Z_+$, there is no $y \in F(x)$ such that $y < y_0$.

b) The condition $0 \in G(x) + Z_+$ is equivalent to $G(x) \cap (-Z_+) \neq \emptyset$.

In order to present an optimality condition for the problem (P), we give a condition for F to be open with a non specific linear rate (in contrast with Theorem 3: in this result the linear rate is known). The main tools of the proof will be Lemma 1, the following lemma due to A. Jourani ([8]) and the methods already used in the previous section.

LEMMA 2. *Let $F : X \rightrightarrows Y$ be a multifunction with closed graph and $(x, y) \in \operatorname{Gr} F$. If F is not metrically regular around (x, y), then there are $(s_n)_{n\in\mathbb{N}}, s_n \downarrow 0$ (which means $s_n \to 0$ and $(s_n) \subset (0,\infty)$), $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$, $x_n \to x$, $y_n \to y$, $z_n \to y$ such that, for every $n \geq 0$, $(x_n, y_n) \notin \operatorname{Gr} F$, $(x_n, z_n) \in \operatorname{Gr} F$ and the function $h : X \times Y \to \mathbb{R}$, $h(u, v) = \|v - y_n\| + s_n(\|u - x_n\| + \|v - z_n\|)$ attains its minimum on $\operatorname{Gr} F$ at the point (x_n, z_n).*

THEOREM 4. *Let $F : X \rightrightarrows Y$ be a multifunction with closed graph and $(x, y) \in \operatorname{Gr} F$. Suppose that there exist $a > 0$ and $r > 0$ such that for each $x' \in B(x, r)$ and $y' \in B(y, r) \cap F(x')$*

$$a \leq \inf\{\|x^*\| \mid x^* \in D^*F(x', y')(y^*), y^* \in S_{Y^*}\}. \tag{7}$$

Then F is open at a linear rate around (x, y).

Proof. Suppose, by contradiction, that F is not open at a linear rate around (x, y). Therefore, by Lemma 1, F is not metrically regular around (x, y) and we can apply Lemma 2: there are $(s_n)_{n\in\mathbb{N}}, s_n \downarrow 0$, $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$, $x_n \to x$, $y_n \to y$, $z_n \to y$ such that, for every $n \geq 0$, $(x_n, y_n) \notin \operatorname{Gr} F$, $(x_n, z_n) \in \operatorname{Gr} F$ and the function $h : X \times Y \to \mathbb{R}$, $h(u, v) = \|v - y_n\| + s_n(\|u - x_n\| + \|v - z_n\|)$ attains its minimum on $\operatorname{Gr} F$ at the point (x_n, z_n). Taking into account the convergence of the sequences (s_n), (x_n), (y_n) and (z_n) we can find $n \in \mathbb{N}$ such that $\|y_n - z_n\| + \|z_n - y\| < r$, $\|y_n - z_n\| + \|x_n - x\| < r$ and $s_n < a/(a + 1)$. For this n, since $y_n \neq z_n$ we can find an $\varepsilon \in (0, \min(\|y_n - z_n\|, a/(a + 1) - s_n)$. For this ε we apply Proposition 1 to the Lipschitz function h. There exist $(x_{1n}, z_{1n}) \in B((x_n, z_n), \varepsilon)$, $(x_{2n}, z_{2n}) \in B((x_n, z_n), \varepsilon) \cap \operatorname{Gr} F$, $(x^*_{1n}, z^*_{1n}) \in \partial h(x_{1n}, z_{1n})$, $(x^*_{2n}, z^*_{2n}) \in N(\operatorname{Gr} F, (x_{2n}, z_{2n}))$ such that

$$\max(\|x^*_{1n} + x^*_{2n}\|, \|z^*_{1n} + z^*_{2n}\|) < \varepsilon.$$

Since $\varepsilon < \|y_n - z_n\|$ and $\|z_{1n} - z_n\| < \varepsilon$, it follows that $y_n \neq z_{1n}$ and this implies that

$$(x^*_{1n}, z^*_{1n}) \in \{0_{X^*}\} \times S_{Y^*} + s_n(U_{X^*} \times \{0_{Y^*}\} + \{0_{X^*}\} \times U_{Y^*});$$

whence $\|x^*_{1n}\| \leq s_n$ and $\|z^*_{1n}\| > 1 - s_n$. Taking into account the inequalities $\|z^*_{1n}\| - \|z^*_{2n}\| \leq \|z^*_{1n} + z^*_{2n}\| < \varepsilon$, we obtain that $\|z^*_{2n}\| > 1 - s_n - \varepsilon > 0$ so, $z^*_{2n} \neq 0$. Obviously, $x^*_{2n}/\|z^*_{2n}\| \in D^*F(x_{2n}, z_{2n})(-z^*_{2n}/\|z^*_{2n}\|)$ and from $\|x^*_{2n}\| < \varepsilon + s_n$, we have

$$\|x^*_{2n}\| / \|z^*_{2n}\| < (\varepsilon + s_n)/(1 - \varepsilon - s_n) < a.$$

Moreover,

$$\begin{aligned}\|x_{2n}-x\| &\leq \|x_{2n}-x_n\|+\|x_n-x\| \\ &< \varepsilon+\|x_n-x\|<\|y_n-z_n\|+\|x_n-x\|<r\end{aligned}$$

and

$$\begin{aligned}\|z_{2n}-y\| &\leq \|z_{2n}-y_n\|+\|y_n-y\| \\ &< \varepsilon+\|y_n-y\|<\|y_n-z_n\|+\|y_n-y\|<r,\end{aligned}$$

which is in contradiction with (7). □

We use the above result to obtain the following corollary.

Corollary 1. *Let X be a Banach space, Y be a finite dimensional normed vector space, $F : X \rightrightarrows Y$ be a multifunction with closed graph and $(x, y) \in \operatorname{Gr} F$. Suppose that:*
a) $0 \in D^*F(x, y)(y^*)$ *implies* $y^* = 0$;
b) ∂ *satisfies the stability property* (P8).
Then the image from F of any neighborhood of x is a neighborhood of y.

Proof. We show that the relation (7) holds. Suppose the contrary and take $a = 1/n, r = 1/n$; we obtain that there are $x_n \to x, y_n \to y$, $y_n \in F(x_n)$, and $x_n^* \to 0$, $y_n^* \in S_{Y^*}$ such that $x_n^* \in D^*F(x_n, y_n)(y_n^*)$. Since (y_n^*) is a bounded sequence, eventually taking a subsequence, we find an element $y^* \in Y$ such that $y_n^* \to y^* \in S_{Y^*}$. Since

$$(x_n^*, -y_n^*) \in N(\operatorname{Gr} F, (x_n, y_n)) = \partial I_{\operatorname{Gr} F}(x_n, y_n),$$

from the stability property we obtain that

$$(0, -y^*) \in \partial I_{\operatorname{Gr} F}(x, y) = N(\operatorname{Gr} F, (x, y))$$

and $y^* \neq 0$, which is a contradiction. We can apply now Theorem 4 and the conclusion follows. □

Remark 4. If F is convex, in Corollary 1 the condition (a) is equivalent to the condition

$$DF(x, y)(X) = Y. \tag{8}$$

Indeed, if (8) holds, take $(0, -y^*) \in N(\operatorname{Gr} F, (x, y))$. Since $\operatorname{Gr} F$ is closed and convex, $0^*(u) - y^*(v) \leq 0$ for all $(u, v) \in T(\operatorname{Gr} F, (x, y))$, i.e., $-y^*(v) \leq 0$ for all $v \in Y$, whence $y^* = 0$. The converse is also true: if not, we use a separation theorem to find an $y^* \in Y^*$, $y^* \neq 0$ such that $y^*(v) \leq 0$ for all $v \in DF(x, y)(X)$, i.e., $0 \in D^*F(x, y)(-y^*)$, a contradiction. Taking into account the surjectivity condition (8), one can see the links with Banach open principle (see also [2]).

We can use the proof of Corollary 1 with few differences and Theorem 2 instead of Theorem 4 to obtain the next corollary.

Corollary 2. *Let X be a Banach space, Y be a finite dimensional space, $F : X \rightrightarrows Y$ be a multifunction with closed graph and $x \in \operatorname{Dom} F$ such that $F(x)$ is a bounded set. Suppose that:*

a) *there exists $V \in \mathcal{V}(x)$ such that $V \subset \operatorname{Dom} F$, F is usc on V;*

b) *if $y \in F(x), 0 \in D^*F(x,y)(y^*)$, then $y^* = 0$;*

c) *∂ satisfies the stability property* (P8).

Then the image by F of any neighborhood of x contains a neighborhood of $F(x)$.

We give now a necessary optimality condition for the problem (P).

Theorem 5. *Let (x_0, y_0) be a weak local minimum point for the problem* (P). *Suppose that ∂ satisfies the stability property* (P8).

Then, for every $z_0 \in G(x_0) \cap (-Z_+)$, there are $y^* \in Y^*$ and $z^* \in Z^*, (y^*, z^*) \neq (0,0)$ such that $y^*(v) + z^*(w) \geq 0$ for every $(v,w) \in D(F \times G)(x_0, y_0, z_0)(X)$.

Proof. Let $z_0 \in G(x_0) \cap (-Z_+)$ and U be the neighborhood of x_0 given by definition of weak local minimum. Suppose that $(F \times G)(U)$ is a neighborhood of (y_0, z_0). Therefore, there exists a neighborhood V of y_0 such that

$$V \times \{z_0\} \subset (F \times G)(U).$$

Then, for every $y \in V$, there exists $x \in U$ such that $y \in F(x)$ and $z_0 \in G(x) \cap (-Z_+)$ and, from the Definition 5, we obtain that $y - y_0 \in Y \backslash (-\operatorname{int} Y_+)$, so $V - y_0 \subset Y \backslash (-\operatorname{int} Y_+)$, which means that there is a neighborhood V_0 of 0, such that $V_0 \subset Y \backslash (-\operatorname{int} Y_+)$, whence $V_0 \cap (-\operatorname{int} Y_+) = \emptyset$. Let $q \in \operatorname{int} Y_+$; since V_0 is absorbing, we can find an $\varepsilon \in (0,1)$, with $-\varepsilon q \in V_0$; but $0 \in Y_+$ and the convexity of Y_+ ensures that $\varepsilon q \in \operatorname{int} Y_+$, whence

$$-\varepsilon q \in V_0 \cap (-\operatorname{int} Y_+).$$

This contradiction proves that $(F \times G)(U)$ is not a neighborhood of (y_0, z_0). For the multifunction $F \times G$ the conclusion of Corollary 1 is false, so the hypothesis a) of this corollary is not satisfied (the others are true). We obtain that there exist $y^* \in Y^*$ and $z^* \in Z^*, (y^*, z^*) \neq (0,0)$ such that

$$0 \in D^*(F \times G)(x_0, y_0, z_0)(y^*, z^*)$$

which means

$$(0, -y^*, -z^*) \in N(\operatorname{Gr}(F \times G), (x_0, y_0, z_0)).$$

Consider now an element $(v,w) \in D(F \times G)(x_0, y_0, z_0)(X)$; there is an $u \in X$, such that $(u, v, w) \in T(\operatorname{Gr}(F \times G), (x_0, y_0, z_0))$, so by definitions, $y^*(v) + z^*(w) \geq 0$. $\square$

Remark 5. It is interesting to observe that in [4] and [10] a similar conclusion is obtained (of course, with other coderivatives) using convexity conditions. In our result we did not use any convexity conditions.

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