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ON A CLASS OF ANALYTIC FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY CONVOLUTION

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Abstract. Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $b_n > 0$ be a fixed analytic function defined on $\Delta = \{z; |z| < 1\}$. In the present investigation, we introduce the class of functions $f = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ satisfying

$$\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) < \alpha \quad (z \in \Delta; 1 < \alpha < 3/2)$$

and obtain the coefficient inequality, coefficient estimate, distortion theorem, and a closure theorem. Also we consider a radius problem. Our result contains several new results as special cases.

MSC 2000. 34C45.

Key words. Starlike function, Ruscheweyh derivative, Salagean derivative, convolution, positive coefficients, coefficient inequality, distortion theorem, radius problem.

1. INTRODUCTION AND DEFINITIONS

Let T be the class of all analytic univalent functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0; z \in \Delta = \{z; |z| < 1\}).$$

A function $f(z) \in T$ is called a function with negative coefficients. The subclass of T consisting of starlike functions of order α, denoted by $TS^*(\alpha)$, is studied by Silverman [6]. Several other class of starlike functions with negative coefficients were studied; e.g., see [1]. For two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the convolution (or Hadamard product) of f and g, denoted by $f * g$ or $(f * g)(z)$, is defined to be function $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$. Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be a fixed analytic function in Δ with $b_n > 0$, $(n \geq 2)$. Using convolution, Ali *et al.* [2] (see also [4]) have studied a more general class of multivalent functions which includes the class $TS_g(\alpha)$ defined by

$$TS_g^*(\alpha) = \left\{ f \in T : \Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) > \alpha \quad (0 \leq \alpha < 1; z \in \Delta) \right\}.$$

Ravichandran and Sivaprasad Kumar [5] have studied a similar class of meromorphic functions. Note that several well-known subclasses of functions are special cases of the class $TS_g^*(\alpha)$ for suitable choices of $g(z)$. When $g(z) = z/(1-z)$, the class $TS_g^*(\alpha)$ is the class $TS^*(\alpha)$ of starlike functions with negative coefficients of order α introduced and studied by Silverman [6]. When

$g(z) = z/(1-z)^2$, the class $TS^*_g(\alpha)$ is the class of convex functions with negative coefficients of order α introduced and studied by Silverman [6]. The class $T_\lambda(\alpha)$ studied by Ahuja [1] is a special case of $TS^*_g(\alpha)$ when $g(z) = z/(1-z)^{\lambda+1}$. Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ with $f(0) = 0 = f'(0) - 1$. The class $M(\alpha)$ defined by

$$M(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \quad (1 < \alpha < 3/2; z \in \Delta) \right\}$$

was investigated by Uralegaddi *et al.* [7]. A subclass of $M(\alpha)$ was recently investigated by Owa and Srivastava [3].

In this paper, we introduce a more general class $PM_g(\alpha)$ of analytic function with positive coefficient motivated by $M(\alpha)$ and the earlier work of Ali *et al.* [2]. For the newly defined class $PM_g(\alpha)$, we obtain the coefficient inequality, coefficient estimate, distortion theorem, and a closure theorem. Also we compute the radius of starlikeness of order β and the radius of convexity of order β for the functions in the class $PM_g(\alpha)$. Our result contains several results as special cases.

Definition 1. Let P be the class of all analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{1}$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n > 0) \tag{2}$$

be a fixed analytic function in Δ. Define the class $PM_g(\alpha)$ by

$$PM_g(\alpha) = \left\{ f \in P : \Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) < \alpha \quad (1 < \alpha < 3/2; z \in \Delta). \right\}$$

When $g(z) = z/(1-z)$, the class $PM_g(\alpha)$ reduces to the subclass $PM(\alpha) = P \cap M(\alpha)$. When $g(z) = z/(1-z)^{\lambda+1}$, the class $PM_g(\alpha)$ reduces to the following class $P_\lambda(\alpha)$

$$P_\lambda(\alpha) = \left\{ f \in P : \Re\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}\right) < \alpha, \quad (\lambda > -1, 1 < \alpha < 3/2; z \in \Delta) \right\},$$

where D^λ denotes the Ruscheweyh derivative of order λ. When $g(z) = z + \sum_{n=2}^{\infty} n^m z^n$, the class of function $PM_g(\alpha)$ reduces to the class $PM_m(\alpha)$ where

$$PM_m(\alpha) = \left\{ f \in P : \Re\left(\frac{z(\mathcal{D}^m f(z))'}{\mathcal{D}^m f(z)}\right) < \alpha \quad (1 < \alpha < 3/2; m \geq 0; z \in \Delta) \right\},$$

where $\mathcal{D}^m$ denotes the Salagean derivative of order m. Also we have

$$PM(\alpha) \equiv P_0(\alpha) \equiv PM_0(\alpha).$$

2. COEFFICIENT INEQUALITIES

Throughout the paper, we assume that the function $f(z)$ is given by the equation (1) and $g(z)$ is given by (2). We first prove a necessary and sufficient condition for functions to be in the class $PM_g(\alpha)$ in the following:

THEOREM 1. *A function $f \in PM_g(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n b_n \leq \alpha - 1 \quad (1 < \alpha < 3/2). \tag{3}$$

Proof. If $f \in PM_g(\alpha)$, then (3) follows from

$$\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) < \alpha$$

by letting $z \to 1-$ through real values. To prove the converse, assume that (3) holds. Then by making use of (3), we obtain

$$\left|\frac{z(f*g)'(z)-(f*g)(z)}{z(f*g)'(z)-(2\alpha-1)(f*g)(z)}\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n b_n}{2(\alpha-1)-\sum_{n=2}^{\infty}[n-(2\alpha-1)]a_n b_n} \leq 1$$

or, equivalently, $f \in PM_g(\alpha)$. □

COROLLARY 1. *A function $f \in P_\lambda(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n B_n(\lambda) \leq \alpha - 1 \quad (1 < \alpha < 3/2),$$

where

$$B_n(\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}{(n-1)!}. \tag{4}$$

COROLLARY 2. *A function $f \in PM_m(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n n^m \leq \alpha - 1 \quad (1 < \alpha < 3/2).$$

Our next Theorem gives an estimate for the coefficient of functions in the class $PM_g(\alpha)$.

THEOREM 2. *If $f \in PM_g(\alpha)$, then*

$$a_n \leq \frac{\alpha-1}{(n-\alpha)b_n}$$

with the equality only for functions of the form

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)b_n}z^n.$$

Proof. Let $f \in PM_g(\alpha)$. By making use of the inequality (3) for $f \in PM_g(\alpha)$, we have

$$(n-\alpha)a_n b_n \leq \sum_{n=2}^{\infty}(n-\alpha)a_n b_n \leq \alpha - 1$$

or $a_n \leq \frac{\alpha-1}{(n-\alpha)b_n}$. Clearly for

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)b_n} z^n \in PM_g(\alpha),$$

we have $a_n = \frac{\alpha-1}{(n-\alpha)b_n}$. □

COROLLARY 3. *If* $f \in P_\lambda(\alpha)$, *then*

$$a_n \leq \frac{\alpha-1}{(n-\alpha)B_n(\lambda)}$$

with the equality only for functions of the form

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)B_n(\lambda)} z^n,$$

where $B_n(\lambda)$ *is given by* (4).

COROLLARY 4. *If* $f \in PM_m(\alpha)$, *then*

$$a_n \leq \frac{\alpha-1}{(n-\alpha)n^m}$$

with the equality only for functions of the form

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)n^m} z^n.$$

3. GROWTH THEOREM

We now prove the growth theorem for the functions in the class $PM_g(\alpha)$.

THEOREM 3. *If* $f \in PM_g(\alpha)$, *then*

$$r - \frac{\alpha-1}{(2-\alpha)b_2} r^2 \leq |f(z)| \leq r + \frac{\alpha-1}{(2-\alpha)b_2} r^2, \quad |z| = r < 1,$$

provided $b_n \geq b_2$. *The result is sharp for*

$$f(z) = z + \frac{\alpha-1}{(2-\alpha)b_2} z^2.$$

Proof. By making use of the inequality (3) for $f \in PM_g(\alpha)$ together with

$$(2-\alpha)b_2 \leq (n-\alpha)b_n,$$

we obtain

$$b_2(2-\alpha)\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty}(n-\alpha)a_n b_n \leq \alpha - 1$$

or

$$\sum_{n=2}^{\infty} a_n \leq \frac{\alpha-1}{(2-\alpha)b_2}. \tag{5}$$

By using (5) for the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in PM_g(\alpha)$, we have

$$\begin{aligned} |f(z)| &\leq r + \sum_{n=2}^{\infty} a_n r^n \quad (|z| = r) \\ &\leq r + r^2 \sum_{n=2}^{\infty} a_n \\ &\leq r + r^2 \frac{\alpha-1}{(2-\alpha)b_2} \end{aligned}$$

and similarly we have

$$|f(z)| \geq r - r^2 \frac{\alpha-1}{(2-\alpha)b_2}\,.$$

□

COROLLARY 5. *If $f \in P_\lambda(\alpha)$, then*

$$r - \frac{\alpha-1}{(2-\alpha)(\lambda+1)} r^2 \leq |f(z)| \leq r + \frac{\alpha-1}{(2-\alpha)(\lambda+1)} r^2 \quad (|z| = r).$$

The result is sharp for

$$f(z) = z + \frac{\alpha-1}{(2-\alpha)(\lambda+1)} z^2.$$

COROLLARY 6. *If $f \in PM_m(\alpha)$, then*

$$r - \frac{\alpha-1}{(2-\alpha)2^m} r^2 \leq |f(z)| \leq r + \frac{\alpha-1}{(2-\alpha)2^m} r^2 \quad (|z| = r).$$

The result is sharp for

$$f(z) = z + \frac{\alpha-1}{(2-\alpha)2^m} z^2.$$

4. CLOSURE THEOREMS

Let the functions $F_k(z)$ be given by

$$F_k(z) = z + \sum_{n=2}^{\infty} f_{n,k} z^n \quad (k = 1, 2, ..., m). \tag{6}$$

We shall now prove the following closure theorems for the class $PM_g(\alpha)$.

THEOREM 4. *Let the function $F_k(z)$ defined by* (6) *be in the class $PM_g(\alpha)$ for every $k = 1, 2, ..., m$. Then the function $f(z)$ defined by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0)$$

belongs to the class $PM_g(\alpha)$, where $a_n = \frac{1}{m}\sum_{k=1}^{m} f_{n,k}$ $(n = 1, 2, \dots)$.

Proof. Since $F_k(z) \in PM_g(\alpha)$, it follows from Theorem 1 that

$$\sum_{n=2}^{\infty}(n-\alpha)g_n f_{n,k} \le \alpha - 1 \tag{7}$$

for every $k = 1, 2, .., m$. Hence

$$\begin{aligned}\sum_{n=2}^{\infty}(n-\alpha)g_n a_n &= \sum_{n=2}^{\infty}(n-\alpha)g_n\left(\frac{1}{m}\sum_{k=1}^{m} f_{n,k}\right)\\ &= \frac{1}{m}\sum_{k=1}^{m}\left(\sum_{n=2}^{\infty}(n-\alpha)g_n f_{n,k}\right)\\ &\le \alpha - 1.\end{aligned}$$

By Theorem 1, it follows that $f(z) \in PM_g(\alpha)$. □

Theorem 5. *The class $PM_g(\alpha)$ is closed under convex linear combination.*

Proof. Let the function $F_k(z)$, $k = 1, 2$, given by (6) be in the class $PM_g(\alpha)$. Then it is enough to show that the function

$$H(z) = \lambda F_1(z) + (1-\lambda)F_2(z) \quad (0 \le \lambda \le 1)$$

is also in the class $PM_g(\alpha)$. Since for $0 \le \lambda \le 1$

$$H(z) = z + \sum_{n=1}^{\infty}[\lambda f_{n,1} + (1-\lambda)f_{n,2}],$$

we observe that

$$\begin{aligned}&\sum_{n=2}^{\infty}(n-\alpha)g_n[\lambda f_{n,1} + (1-\lambda)f_{n,2}]\\ =\ & \lambda\sum_{n=2}^{\infty}(n-\alpha)g_n f_{n,1} + (1-\lambda)\sum_{n=2}^{\infty}(n-\alpha)g_n f_{n,2}\\ \le\ & \alpha - 1.\end{aligned}$$

By Theorem 1, we have $H(z) \in PM_g(\alpha)$. □

Theorem 6. *Let $F_1(z) = z$ and $F_n(z) = z + \frac{\alpha-1}{(n-\alpha)g_n}z^n$ for n=2,3,... . Then $f(z) \in PM_g(\alpha)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=1}^{\infty}\lambda_n F_n(z)$ where $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty}\lambda_n = 1$.*

Proof. Let

$$\begin{aligned}f(z) &= \sum_{n=1}^{\infty}\lambda_n F_n(z)\\ &= z + \sum_{n=2}^{\infty}\frac{\lambda_n(\alpha-1)}{(n-\alpha)g_n}z^n.\end{aligned}$$

Then

$$\sum_{n=2}^{\infty} \frac{\lambda_n(\alpha-1)}{(n-\alpha)g_n} \frac{(n-\alpha)g_n}{(\alpha-1)} = \sum_{n=2}^{\infty} \lambda_n = 1-\lambda_1 \leq 1.$$

By Theorem 1, we have $f(z) \in PM_g(\alpha)$.

Conversely, let $f(z) \in PM_g(\alpha)$. From Theorem 2, we have

$$f_n \leq \frac{\alpha-1}{(n-\alpha)g_n} \quad \text{for} \quad n = 2, 3, \dots.$$

Therefore we may take

$$\lambda_n = \frac{(n-\alpha)g_n f_n}{\alpha-1} \quad \text{for} \quad n = 2, 3, \dots$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

Then $f(z) = \sum_{n=1}^{\infty} \lambda_n F_n(z)$. □

5. RADIUS PROBLEM

In this section, we find the radius of starlikeness of order β and the radius of convexity of order β for functions in the class $PM_g(\alpha)$.

THEOREM 7. *If $f \in PM_g(\alpha)$ $(1 < \alpha \leq 3/2)$, then f is starlike of order β $(0 \leq \beta < 1)$ in $|z| < r(\beta, \alpha, g)$ where*

$$r(\beta, \alpha, g) = \inf_{n \geq 2} \left[\frac{(1-\beta)(n-\alpha)}{(\alpha-1)(n-\beta)} b_n \right]^{1/(n-1)}.$$

Proof. It is enough to show that

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} a_n |z|^{n-1} < 1 \tag{8}$$

which will imply that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \beta.$$

The inequality (8) follows if

$$\frac{n-\beta}{1-\beta} a_n |z|^{n-1} \leq \frac{n-\alpha}{\alpha-1} a_n b_n$$

and this proves the result. □

We have the following:

COROLLARY 7. *If $f \in PM_g(\alpha)$ $(1 < \alpha \leq 3/2)$, then f is convex of order β $(0 \leq \beta < 1)$ in $|z| < r(\beta, \alpha, g)$ where*

$$r(\beta, \alpha, g) = \inf_{n \geq 2} \left[\frac{(1-\beta)(n-\alpha)}{n(\alpha-1)(n-\beta)} b_n \right]^{1/(n-1)}.$$

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