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THE SINE FUNCTIONAL EQUATION ON 2-DIVISIBLE GROUPS

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Abstract. We solve the functional equation $f : G \to K$, $f(xy)f(xy^{-1}) = f^2(x) - f^2(y)$ in the case of a group G divisible by 2 and of a field K with $\operatorname{char} K \neq 2$.

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Consider the sine functional equation

$$f : G \to K, \quad f(xy)f(xy^{-1}) = f^2(x) - f^2(y) \tag{1}$$

where G is a group and K is a field of characteristic different from 2.

Several papers deal with the sine equation when G is an abelian group. See, e.g., the monographs [1] and [2] by Aczél and by Aczél and Dhombres for references and results.

THEOREM 1. [2, Theorem 14] *Let K be a quadratically closed field with $\operatorname{char} K \neq 2$. Let G be an abelian group divisible by 2. The general solutions $f : G \to K$ of (1) are given by*

$$f(x) = a\frac{g(x) - g(x^{-1})}{2} \tag{2}$$

and

$$f(x) = \beta(x), \tag{3}$$

where g is a homomorphism from G into the multiplicative group of K, β is a homomorphism from G into the additive group of K and a is an arbitrary element of K.

In the particular cases [1, Theorem p. 137] or [2, Corollary 15], it is known that the general solutions $f : \mathbb{R} \to \mathbb{C}$ or $f : \mathbb{R} \to \mathbb{R}$ of (1) in the class of functions measurable on a proper interval are given by

$$f(x) = a \sinh cx, \quad f(x) = cx$$

and, respectively,

$$f(x) = a \sinh cx, \quad f(x) = a \sin cx, \quad f(x) = cx,$$

where a and c are complex or real constants, respectively.

There are only few results in the literature on the sine functional equation in non-abelian groups. The author has solved in [3] and [4] the equation (1) in a special non-abelian group.

THEOREM 2. [3, Theorem 3] *Let G be a group whose elements are of odd order and K be a quadratically field with* $\operatorname{char} K \neq 0$. *Then the general solutions $f : G \to K$ of* (1) *are given by* (2) *and* (3).

If G is a cyclic group and K is the field of complex numbers, Pl. Kannappan in [5] has proved that for a solution f of the form (2) the homomorphism g is unique.

In this paper we solve the equation (1) in the case when G is a group divisible by 2 and K is supposed to be a field with $\operatorname{char} K \neq 2$, hence this result is a generalization of the results from [2] and [3].

We notice that $f(x) \equiv 0$ is a trivial solution of (1). In the sequel we shall consider only the solutions not identically zero.

If $f : G \to K$ is a solution of the equation (1) then f has the following properties:

$$f(e) = 0\,, \tag{4}$$

where e is the neutral element of G, and

$$f(x) = -f(x^{-1})\,, \quad \forall\, x \in G, \tag{5}$$

that is, f is an odd function.

Indeed, taking $x = y = e$ in (1) we obtain (4) and for $x = e$ in (1) we get

$$f(y)[f(y) + f(y^{-1})] = 0.$$

If $f(y) \neq 0$, then (5) follows from this equality. Setting y^{-1} for y in (1) and taking $x = e$, by $f(y) = 0$ and by (4) we obtain $f(y^{-1}) = 0$. Hence (5) is true for all $x \in G$.

LEMMA 1. *If G is a group, K is a field and $f : G \to K$ is a solution of the equation* (1), *then $A_f(G) = \{x|\ x \in G,\ f(x) = 0\}$ is a normal subgroup of G (see* [3, Lemma 1]*).*

Proof. We will show that

$$\forall\, x, y \in A_f(G) \ \Rightarrow\ xy^{-1} \in A_f(G)\,, \tag{6}$$

i.e. $A_f(G)$ is a subgroup of G.

From (5) we have that $x \in A_f \ \Rightarrow\ x^{-1} \in A_f(G)$. Putting $y = x^2$, $x \in A_f(G)$ in (1) we get $f(x^3)f(x^{-1}) = f^2(x) - f^2(x^2) = 0$, hence $x^2 \in A_f$.

From (1) we get $f(yx)f(yx^{-1}) = f^2(y) - f^2(x) = 0$, for $x, y \in A_f(G)$, therefore $yx^{-1} \in A_f$ implies $(yx^{-1}) = xy^{-1} \in A_f$ and (6) is true, for $yx \in A_f(G)$. Using (1) we obtain

$$f^2(x^{-1}y^{-1}) - f^2(xy^{-1}) = f(x^{-1}y^{-1}xy^{-1})f(x^{-1}y^{-1}yx^{-1}) =$$
$$= f(x^{-1}y^{-1}xy^{-1})f(x^{-2}).$$

But $x^{-1}y^{-1} = (yx)^{-1} \in A_f(G)$ and $x^{-1} \in A_f(G)$, hence $xy^{-1} \in A_f(G)$ and (6) is true in this case, too.

Now we show that $A_f(G)$ is a normal subgroup. To this end, we prove first that

$$xy \in A_f(G) \ \Rightarrow \ yx \in A_f(G)\,. \tag{7}$$

Since $f(xy) = 0$, from (1) we have

$$f^2(x) = f^2(y). \tag{8}$$

Using (1), (5) and (8) we get $f(x^{-1}y^{-1})f(x^{-1}y) = f^2(x^{-1}) - f^2(y) = 0$ and $f(yx)f(yx^{-1}) = f^2(y) - f^2(x) = 0$, hence we have $f(yx)f(x^{-1}y) = 0$ and $f(yx)f(yx^{-1}) = 0$.

If $yx \in A_f(G)$, (7) is true. It remains to prove (7) if $x^{-1}y \in A_f(G)$ and $yx^{-1} \in A_f(G)$.

From (1) we have

$$f^2(y^{-1}) - f^2(yxy^{-1}) = f(xy^{-1})f(x^{-1}y^{-1}) = f(yx^{-1})f(yx) = 0. \tag{9}$$

Using (1), (8) and (9) we obtain

$$f^2(yxy^{-1}) - f^2(x) = f(yxy^{-1}x)f(yxy^{-1}x^{-1}) = 0 \tag{10}$$

and

$$f^2(xy) - f^2(yx) = f(xy^2x)f(xyx^{-1}y^{-1}).$$

Since $xy \in A_f(G)$ we deduce

$$-f^2(yx) = f(xy^2x)f(xyx^{-1}y^{-1}). \tag{11}$$

Because $y^{-1}x = (x^{-1}y)^{-1} \in A_f(G)$ we have

$$f^2(yx) - f^2(y^{-1}x) = f^2(yx) = f(yxy^{-1}x)f(y^2). \tag{12}$$

From (11) and (12) we obtain

$$f^4(yx) = f(yxy^{-1}x^{-1})f(yxy^{-1}x)f(y^2)f(xy^2x)$$

and, in view of (10), this yields $f(yx) = 0$, i.e. $yx \in A_f(G)$ and (7) is true for all $x, y \in G$.

Let $u \in A_f(G)$ and $x \in G$. By (7) we can write $0 = f(u) = f(x^{-1}xu) = f(xux^{-1})$, yielding $xux^{-1} \in A_f(G)$. □

LEMMA 2. *Let G be a 2-divisible group and K be a field. If $f : G \to K$ is a solution of the sine equation, then*

$$f(u) = 0\,, \quad \forall u \in G'. \tag{13}$$

Proof. Interchanging x with y in (1) we obtain

$$f(yx)f(yx^{-1}) = f^2(y) - f^2(x).$$

But $(yx^{-1})^{-1} = xy^{-1}$ and using (5) this equality becomes

$$f(yx)f(xy^{-1}) = f^2(x) - f^2(y).$$

From this equality and (1) it follows that

$$[f(xy) - f(yx)]f(xy^{-1}) = 0, \quad \forall\, x, y \in G. \tag{14}$$

Hence, $f(xy) = f(yx)$ or $f(xy^{-1}) = 0$, for every $x, y \in G$.

If $f(xy) = f(yx)$, replacing x by xy and y by yx in (1), we get

$$f(xyx^{-1}y^{-1})f(xy^2x) = 0.$$

Therefore for any $x, y \in G$ we have three possibilities:

i) $xyx^{-1}y^{-1} \in A_f(G)$;

ii) $xy^2x \in A_f(G)$;

iii) $xy^{-1} \in A_f(G)$.

G being a 2-divisible group there exist $u, v \in G$ such that $u^2 = x$ and $v^2 = y$.

Now for $u, v \in G$ we have three possibilities:

i) $uvu^{-1}v^{-1} \in A_f(G)$. Then $(uvu^{-1}v^{-1})(vu^{-1}v^{-1}u) = uvu^{-2}v^{-1}u = u^2vu^{-2}v^{-1} \in A_f(G)$, which implies $(vu^{-2}v^{-1}u^2)(u^{-2}v^{-1}u^2v) = vu^{-2}v^{-2}u^2v = v^2u^{-2}v^{-2}u^2 \in A_f$ and therefore $yx^{-1}y^{-1}x \in A_f(G)$.

ii) $uv^2u \in A_f(G)$. Then $u^2v^2 \in A_f(G)$, so that $xy \in A_f(G)$. Hence we have $xy(yx)^{-1} \in A_f(G)$ and therefore $xyx^{-1}y^{-1} \in A_f(G)$.

iii) $uv^{-1} \in A_f(G)$. Then $uv^{-1}u^{-1}v \in A_f(G)$, hence this case reduces to case i). □

THEOREM 3. *Let G be a group divisible by 2 such that G' is a 2-divisible subgroup of G and K be a quadratically closed field with* $\operatorname{char} K \neq 2$. *If $f : G \to K$ is a solution of the sine equation* (1), *then f has the form* (2) *or* (3).

Proof. From Lemmas 1 and 2 we have

$$f^2(xu) - f^2(xu^{-1}) = f(xuxu^{-1})f(xu^2x^{-1}) = 0$$

Hence $f^2(xu) = f^2(xu^{-1})$, $\forall x \in G$, $\forall u \in G'$. Setting xu for x we have $f^2(xu^2) = f^2(x)$. Since G' is 2-divisible we have $f^2(xu) = f^2(x)$ for all $x \in G$ and $u \in G'$. □

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