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SOME REMARKS ON DIFFERENTIAL INCLUSIONS WITH STATE CONSTRAINTS

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Abstract. We prove the existence of solutions to lipschitzean integrodifferential inclusions viable in a closed set contained in R^n. Using this result we study the infinitesimal properties of functions which are nonincreasing along all the trajectories of a differential inclusion.

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Key words. Lipschitzean set-valued maps, differential inclusions, viable solutions, generalized derivatives.

1. INTRODUCTION

In the viability theory for differential inclusions there are several papers devoted to the case when the values of the multifunction that define the differential inclusion are not convex. The main assumption is that the multifunction is (locally) lipschitzean in the state variable ([1], [3], [8], [9], etc.). In [3] it is proved the existence of solutions to the differential inclusion $x' \in F(t,x)$ viable in a compact subset of finite dimensional space and continuously depending on an initial point from this subset. This result was extended in [8] to the case of a family of differential inclusions and a family of viability constraints in separable metric spaces. Using the Filippov technique ([6]), the result in [8] contains also a Filippov-Gronwall type inequality.

In this paper we adapt the idea of Goncharov to the more general problem of integrodifferential inclusions. In the particular case of differential inclusions our result improve the (Filippov type) estimations in [8]. In the stability theory of nonlinear systems the second Lyapunov method consists in find a function W with several properties. An essential property is the following monotonicity condition: the map $W \circ x(.)$ is nonincreasing for any admissible trajectory x. In a recent paper ([2]) the situation when the system is described by an autonomous differential inclusion is considered. More exactly, it is given a characterization of the monotonicity condition presented above in terms of contingent derivatives, when W is lower semicontinuous and the multifunction that define the differential inclusion is locally lipschitzean and compact valued. Using our viability result, we extend the result of Bacciotti, Ceragioli and Mazzi ([2]) to the case of differential inclusions whose trajectories are constrained to a given closed set. The paper is organized as follows: in Section 2 we present the notations and definitions to be used in the sequel. Section 3 is devoted to the existence of viable solutions and in Section 4 we provide a characterization of the monotonicity of the Lyapunov function.

2. PRELIMINARIES

In this paper we shall be concerned with the absolutely continuous solutions $x(.) : [0,T] \to R^n$ of the integrodifferential inclusion

$$x' \in F(t,x,V(x)(t)), \quad x(0) = x_0 \in S, \tag{2.1}$$

satisfying state constraints of the form

$$x(t) \in S, \quad \forall t \in [0,T], \tag{2.2}$$

where $F(.,.,.) : [0,T] \times S \times R^n \to \mathcal{P}(R^n)$ (with $\mathcal{P}(R^n)$ we denote the family of all subsets of R^n) is a given set-valued map, $V(.) : C([0,T],R^n) \to C([0,T],R^n)$ is a nonlinear (integral) operator (with $C([0,T],R^n)$ we denote the space of all continuous functions $x : [0,T] \to R^n$ endowed with the norm $||x(.)||_C = \sup_{t\in[0,T]} ||x(t)||$) and $S \subset R^n$ is a given set. Denote by I the interval $[0,T]$, $T > 0$ and by $AC(I,R^n)$ the space of all absolutely continuous functions endowed with the norm $||x(.)||_{AC} = ||x(0)|| + \int_0^T ||x'(t)||\mathrm{d}t$ and by $L^1(I,R^n)$ the space of Lebesgue integrable mappings endowed with the norm $||u(.)||_1 = \int_0^T ||u(t)||\mathrm{d}t$. If $A \subset R^n$ then by $co(A)$ (resp. $\overline{co}(A)$) we denote the convex (resp. closed convex) hull of A. We recall that on $\mathcal{P}(R^n)$ the generalized Hausdorff-Pompeiu metric is defined by

$$\mathrm{d}_H(A,B) = \max\{\mathrm{d}^*(A,B), \mathrm{d}^*(B,A)\}, \quad \mathrm{d}^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $\mathrm{d}(a,B) = \inf_{b\in B} \mathrm{d}(a,b)$.

The contingent cone to the set $Y \subset R^n$ at the point $y \in \overline{Y}$ (the closure of Y) is defined by

$$K_yY = \{v \in R^n; \exists s_m \to 0+,\ \exists v_m \to v :\ x + s_m v_m \in Y\}.$$

The Clarke's tangent cone to the set Y at the point $y \in \overline{Y}$ is defined by

$$C_yY = \{v \in R^n; \forall (x_m, s_m) \to (y, 0+),\ x_m \in Y,\ \exists y_m \in Y :\ \frac{y_m - x_m}{s_m} \to v\}.$$

Finally, we recall that given a function $f : R^n \to R$, its lower right contingent directional derivative at a point $x \in R^n$ and in the direction $w \in R^n$ is defined by

$$\underline{D}_K^+ f(x;w) = \liminf_{h\to 0+, v\to w} \frac{f(x+hv) - f(x)}{h}.$$

In what follows we assume the following hypothesis.

Hypothesis 2.1. i) *$S \subset R^n$ is closed.*

ii) *There exists $M \geq 0$ such that*

$$||V(x_1)(t) - V(x_2)(t)|| \leq M||x_1(t) - x_2(t)||, \quad \forall t \in I, \forall x_1(.), x_2(.) \in C(I,R^n).$$

iii) *$F(.,.,.) : I \times S \times R^n \to \mathcal{P}(R^n)$ has nonempty closed values and for any $x \in S, y \in R^n$ $F(.,x,y)$ is measurable*

iv) *There exists* $L(.) \in L^1(I, R)$ *such that, for all* $t \in I$, *and for all* $x_1, x_2 \in S, y_1, y_2 \in R^n$

$$d_H(F(t, x_1, y_1), F(t, x_2, y_2)) \leq L(t)(||x_1 - x_2|| + ||y_1 - y_2||).$$

v) $F(t, x, y) \subset K_x S \ \forall x \in S, y \in R^n$, *a.e.* $t \in I$.

vi) *There exists* $y(.) \in AC(I, R^n)$ *and* $p(.) \in L^1(I, R)$ *such that*

$$d(y'(t), F(t, g(t), V(g)(t))) \leq p(t) \quad a.e.\ (I),$$

where the measurable function $g(.) : I \to S$ *is such that*

$$||y(t) - g(t)|| = d(y(t), S) \quad a.e.\ (I).$$

We shall use the notations: $L_1(t) = (1 + M)L(t)$, $m(t) = \int_0^t L_1(s)ds$, $t \in I$.

3. VIABLE SOLUTIONS OF LIPSCHITZEAN DIFFERENTIAL INCLUSIONS

The main result of this section is the following.

THEOREM 3.1. *We assume that* Hypothesis 2.1 *is satisfied.*

Then, for any $x_0 \in S$ *there exists* $x(.) \in AC(I, R^n)$ *a solution to the problem* (2.1)–(2.2) *such that for any* $t \in I$ *one has*

$$\int_0^t ||x'(s) - y'(s)||ds \tag{3.1}$$

$$\leq \frac{2}{3}\int_0^t [e^{3(m(t)-m(s))} - 1]p(s)ds + \frac{e^{3m(t)} - 1}{3}[d(y(0), S) + ||y(0) - x_0||].$$

Proof. Define $x_0(t) = x(t)$, $g_0(t) = g(t)$ and $\rho(t) = d(y(0), S) + \int_0^t p(s)ds$, $t \in I$. Denote by $\Pi_S(x) = \{y \in S; ||y - x|| = d(x, S)\}$ the projection of the point x to the set S.

Let us note that according to Theorem 4.3.8 in [5] (see also Proposition 2.2 in [8]) if $F(t, ., .)$ is locally lipschitzean, then the condition

$$F(t, x, y) \subset K_x S, \quad \forall x \in S, y \in R^n, \quad a.e. t \in I$$

is equivalent to

$$F(t, x, y) \subset C_x S, \quad \forall x \in S, y \in R^n, \quad a.e. t \in I.$$

Then, the mapping $h_0(t) = d(x_0(t), S)$, $t \in I$ is absolutely continuous and its derivative satisfies, by Lemma 1 in [9] and Hypothesis 2.1 v)

$$h_0'(t) = \lim_{\delta \to 0+} \frac{1}{\delta}[d(x_0(t) + \delta x_0'(t), S) - d(x_0(t), S)]$$

$$\leq \inf_{g \in \Pi_S(x_0(t))} d(x_0'(t), C_g S) \leq d(x_0'(t), F(t, g_0(t), V(g_0(t))) \leq p(t).$$

Therefore,

$$d(x_0(t), S) = h_0(t) \leq d(x_0(0), S) + \int_0^t d\Big(x_0'(s), F(s, g_0(s), V(g_0)(s))\Big)ds \tag{3.2}$$

$$= \mathrm{d}(y(0), S) + \int_0^t p(s)\mathrm{d}s = \rho_0(t).$$

Applying a well known consequence of the Kuratowski and Ryll-Nardzewski selection theorem (e.g. Theorem 1.14.2 in [1]) there exists a measurable function $f_0(.) : I \to R^n$ such that

$$f_0(t) \in F(t, g_0(t), V(g_0)(t)) \quad a.e.\ (I),$$

$$||x_0'(t) - f_0(t)|| = \mathrm{d}\Big(x_0'(t), F(t, g_0(t), V(g_0)(t))\Big) \quad a.e.\ (I).$$

We define

$$x_1(t) = x_0 + \int_0^t f_0(s)\mathrm{d}s, \quad t \in I$$

and we take $g_1(.) : I \to R^n$ measurable such that

$$||x_1(t) - g_1(t)|| = \mathrm{d}(x_1(t), S) \quad a.e.\,(I).$$

Put

$$\rho_1(t) = \int_0^t \left[\rho_0(s) + \int_0^s p(u)\mathrm{d}u + ||y(0) - x_0||\right] L_1(s)\mathrm{e}^{m(t)-m(s)}\mathrm{d}s,$$

$$\beta_1(t) = L_1(t)\left[\rho_0(t) + \rho_1(t) + \int_0^t p(s)\mathrm{d}s + ||y(0) - x_0||\right].$$

We prove next that, for $t \in I$,

$$||x_1'(t) - x_0'(t)|| \le p(t), \tag{3.3}$$

$$\mathrm{d}(x_1(t), S) \le \rho_1(t), \tag{3.4}$$

$$\mathrm{d}(x_1'(t), F(t, g_1(t), V(g_1)(t))) \le \beta_1(t). \tag{3.5}$$

The inequality (3.3) is obvious. From Hypothesis 2.1 v), as in the proof of (3.2), we obtain

$$\mathrm{d}(x_1(t), S) \le \int_0^t \mathrm{d}\Big(x_1'(s), F(s, g_1(s), V(g_1)(s))\Big)\mathrm{d}s. \tag{3.6}$$

On the other hand, one has

$$\mathrm{d}(x_1'(t), F(t, g_1(t), V(g_1)(t))) = \mathrm{d}\Big(f_0(t), F(t, g_1(t), V(g_1)(t))\Big) \le \tag{3.7}$$

$$\le L(t)[||g_1(t) - g_0(t)|| + M||g_1(t) - g_0(t)||] \le L_1(t)[||g_1(t) - x_1(t)|| +$$

$$||x_1(t) - x_0(t)|| + ||x_0(t) - g_0(t)||] \le L_1(t)[d(x_1(t), S) + ||x_1(t) - x_0(t)|| + \rho_0(t)]$$

$$\le L_1(t)\left[d(x_1(t), S) + ||y(0) - x_0|| + \int_0^t p(s)\mathrm{d}s + \rho_0(t)\right].$$

Therefore, from (3.6) it follows

$$\mathrm{d}(x_1(t), S) \le \int_0^t L_1(s)\left[\mathrm{d}(x_1(s), S) + ||y(0) - x_0|| + \int_0^s p(u)\mathrm{d}u + \rho_0(s)\right]\mathrm{d}s$$

and applying the Gronwall inequality we obtain

$$\mathrm{d}(x_1(t), S) \le \rho_1(t),$$

i.e. (3.4) and from (3.7) we get (3.5).

Define, now, the sequences of continuous mappings $\rho_k, \beta_k : I \to R^n$ by the following recursive relations

$$\rho_{k+1}(t) = \int_0^t \left[\rho_k(s) + \int_0^s \beta_k(u)\mathrm{d}u\right] L_1(s)\mathrm{e}^{m(t)-m(s)}\mathrm{d}s, \tag{3.8}$$

$$\beta_{k+1}(t) = L_1(t)\left[\rho_k(t) + \rho_{k+1}(t) + \int_0^t \beta_k(s)\mathrm{d}s\right]. \tag{3.9}$$

We claim that there exist sequences $x_k(.), g_k(.) : I \to R^n$ with $x_k(0) = x_0$, $x_k(.)$ is absolutely continuous, $g_k(.)$ is measurable and such that, for all $t \in I$

$$||x'_{k+1}(t) - x'_k(t)|| = \mathrm{d}\Big(x'_k(t), F(t, g_k(t), V(g_k)(t))\Big) \le \beta_k(t), \tag{3.10}$$

$$||x_k(t) - g_k(t)|| = \mathrm{d}(x_k(t), S) \le \rho_k(t), \tag{3.11}$$

Suppose that the mappings $x_k(.), g_k(.)$ are already constructed for some $k \ge 1$. Let $f_k(.) : I \to R^n$ be measurable such that

$$f_k(t) \in F(t, g_k(t), V(g_k)(t)) \quad a.e.\ (I),$$

$$||x'_k(t) - f_k(t)|| = \mathrm{d}\Big(x'_k(t), F(t, g_k(t), V(g_k)(t))\Big) \quad a.e.\ (I).$$

We define

$$x_{k+1}(t) = x_0 + \int_0^t f_k(s)\mathrm{d}s, \quad t \in I$$

and we take $g_{k+1}(.) : I \to R^n$ measurable such that

$$||x_{k+1}(t) - g_{k+1}(t)|| = \mathrm{d}(x_{k+1}(t), S), \quad a.e.\,(I).$$

From Hypothesis 2.1 v), as in the proof of (3.2), we obtain

$$\mathrm{d}(x_{k+1}(t), S) \le \int_0^t \mathrm{d}\Big(x'_{k+1}(s), F(s, g_{k+1}(s), V(g_{k+1})(s))\Big)\mathrm{d}s. \tag{3.12}$$

At the same time we have

$$\begin{aligned} &\mathrm{d}\Big(x'_{k+1}(t), F(t, g_{k+1}(t), V(g_{k+1})(t))\Big) \le L(t)(1+M)||g_{k+1}(t) - g_k(t)|| \\ &\le L_1(t)[||g_{k+1}(t) - x_{k+1}(t)|| + ||x_{k+1}(t) - x_k(t)|| + ||x_k(t) - g_k(t)||] \\ &\le L_1(t)\left(\mathrm{d}(x_{k+1}(t), S) + \int_0^t \beta_k(s)\mathrm{d}s + \rho_k(t)\right). \end{aligned} \tag{3.13}$$

Therefore, from (3.12) it follows

$$\mathrm{d}(x_{k+1}(t), S) \le \int_0^t L_1(s)\left[\mathrm{d}(x_{k+1}(s), S) + \int_0^s \beta_k(u)\mathrm{d}u + \rho_k(s)\right]\mathrm{d}s$$

and applying the Gronwall inequality we get, for all $t \in I$

$$\mathrm{d}(x_{k+1}(t), S) \leq \rho_{k+1}(t)$$

and from (3.12) we infer, for all $t \in I$

$$\mathrm{d}\Big(x'_{k+1}(t), F(t, g_{k+1}(t), V(g_{k+1})(t))\Big) \leq \beta_{k+1}(t).$$

From (3.8) and (3.9) we deduce that for $k \geq 2$

$$\rho'_k(t) = \beta_k(t), \quad a.e.\ (I).$$

Hence, from (3.8) we find that

$$\rho_{k+1}(t) = 2\int_0^t \rho_k(s) L_1(s) \mathrm{e}^{m(t)-m(s)} \mathrm{d}s, \quad t \in I, k \geq 1. \tag{3.14}$$

According to the definition of $\rho_1(.)$ one may write

$$\rho_1(t) = \int_0^t \left[\rho_0(s) + \int_0^s p(u)\mathrm{d}u + ||y(0) - x_0||\right] L_1(s)\mathrm{e}^{m(t)-m(s)}\mathrm{d}s \leq$$

$$\left[\mathrm{d}(y(0), S) + 2\int_0^T p(s)\mathrm{d}s + ||y(0) - x_0||\right] (\mathrm{e}^{m(T)} - 1) =: m_0.$$

Then, by induction we find that

$$\rho_k(t) \leq m_0 \mathrm{e}^{m(t)} \frac{(2m(t))^{k-1}}{(k-1)!}, \quad k \geq 1, t \in I. \tag{3.15}$$

The estimation in (3.15) implies the convergence of the series $\sum_{k\geq 1} \rho_k(t)$, $t \in I$. Taking into account (3.10) we conclude that the sequence $x_k(.)$ is Cauchy in the Banach space $AC(I, R^n)$ and therefore it converges to some function $x(.) \in AC(I, R^n)$, $x(0) = x_0$. From (3.11) $x(t) \in S$ $\forall t \in I$ and $g_k(t) \to x(t)$ as $k \to \infty$ $\forall t \in I$. Using (3.11) one may write successively

$$\begin{aligned}
& \int_0^t \mathrm{d}\Big(x'(s), F(t, x(s), V(x)(s))\Big)\mathrm{d}s \\
\leq\ & \int_0^t \Big[\mathrm{d}(x'_k(s), x'(s)) + \mathrm{d}\Big(x'_k(s), F(s, g_k(s), V(g_k)(s))\Big) \\
+\ & \mathrm{d}_H\Big(F(s, g_k(s), V(g_k)(s)), F(s, x_k(s), V(x_k)(s))\Big)\Big]\mathrm{d}s \\
\leq\ & \int_0^t \Big[||x'_k(s) - x'(s)|| + L_1(s)||x_k(s) - g_k(s)|| + \\
+\ & \mathrm{d}\Big(x'_k(s), F(t, g_k(s), V(g_k)(s))\Big)\Big]\mathrm{d}s \leq ||x_k(.) - x(.)||_{AC(I,R^n)} \\
+\ & \int_0^t L(s)||x_k(s) - g_k(s)||\mathrm{d}s + \rho_k(t).
\end{aligned}$$

Passing with $k \to \infty$, from the last estimation, since $F(.,.,.)$ has closed values we infer that $x'(t) \in F(t, x(t), V(x)(t))$ a.e. (I), i.e. $x(.)$ is solution to problem (2.1)–(2.2).

It remains to prove the estimation in (3.1). We have, for $t \in I$

$$\int_0^t ||x'(s) - y'(s)|| \mathrm{d}s \leq ||x_k(.) - x(.)||_{AC(I,R^n)} + \sum_{k \geq 1} \rho_k(t).$$

Therefore, $\int_0^t ||x'(s) - y'(s)|| \mathrm{d}s \leq \Sigma(t)$, $\forall t \in I$, where $\Sigma(t) = \sum_{k \geq 1} \rho_k(t)$.

By (3.15) the mapping $\Sigma(.)$ is the solution of the integral equation

$$\Sigma(t) = \rho_1(t) + \int_0^t 2\Sigma(s)L(s)\mathrm{e}^{m(t)-m(s)} \mathrm{d}s,$$

or, equivalently $\Sigma(.)$ is solution to the following Cauchy problem associated to an affine scalar equation

$$\Sigma'(t) = 3\Sigma(t)L(t) + \rho_1'(t) - \rho_1(t)L(t), \quad \Sigma(0) = 0. \tag{3.16}$$

An elementary computation shows that

$$\Sigma(t) = \frac{2}{3}\int_0^t [\mathrm{e}^{3(m(t)-m(s))} - 1]\rho_1'(s)\mathrm{d}s + \frac{1}{3}\rho_1(t), \quad t \in I,$$

which can be rewrite in the form

$$\Sigma(t) = \frac{2}{3}\int_0^t [\mathrm{e}^{3(m(t)-m(s))} - 1]p(s)\mathrm{d}s + \frac{\mathrm{e}^{3m(t)} - 1}{3}[d(y(0), S) + ||y(0) - x_0||]$$

and the proof of theorem is complete. □

4. MONOTONICITY CONDITIONS FOR DIFFERENTIAL INCLUSIONS

In what follows we consider the next version of problem (2.1)–(2.2) in which F does not depends on the first and third variable.

$$x' \in F(x), \quad x(0) = x_0 \in S, \tag{4.1}$$

$$x(t) \in S, \quad \forall t \in I := [0, T]. \tag{4.2}$$

In this case Hypothesis 2.1 becomes:

HYPOTHESIS 4.1. i) *$S \subset R^n$ is closed.*

ii) *$F(.) : S \to \mathcal{P}(R^n)$ has nonempty and closed values.*

iii) *There exists $L > 0$ such that, for all $x, y \in S$*

$$\mathrm{d}_H(F(x), F(y)) \leq L||x - y||.$$

iv) *$F(x) \subset K_x S$ $\forall x \in S$.*

v) *There exists $y(.) \in AC(I, R^n)$ and $p(.) \in L^1(I, R)$ such that*

$$\mathrm{d}(y'(t), F(g(t))) \leq p(t) \quad a.e.\ (I),$$

where the measurable function $g(.) : I \to S$ is such that

$$||y(t) - g(t)|| = \mathrm{d}(y(t), S) \quad a.e.\ (I).$$

The corresponding variant of Theorem 3.1 is:

THEOREM 4.2. *Assume that* Hypothesis 4.1 *is satisfied.*
Then, for any $x_0 \in S$ there exists $x(.) \in AC(I, R^n)$ a solution to the problem (4.1)–(4.2) *such that for any $t \in I$ one has*

$$\int_0^t ||x'(s) - y'(s)||\mathrm{d}s \leq \frac{2}{3}\int_0^t [\mathrm{e}^{3L(t-s)} - 1]p(s)\mathrm{d}s + \tag{4.3}$$
$$+\frac{\mathrm{e}^{3Lt} - 1}{3}[\mathrm{d}(y(0), S) + ||y(0) - x_0||].$$

REMARK 4.3. We note that according to Theorem 3.1 in [8] under similar hypothesis, for any $\epsilon > 0$ there exists a mapping $x_\epsilon(.) \in AC(I, R^n)$ solution to (4.1)–(4.2) such that for all $t \in I$

$$\int_0^t ||x'_\epsilon(s) - y'(s)||\mathrm{d}s \leq \tag{4.4}$$
$$\int_0^t \phi(L(t-s))p(s)\mathrm{d}s + \phi(Lt)[\epsilon + \mathrm{d}(y(0), S) + ||y(0) - x_0||],$$

where $\phi : [0, \infty) \to [0, \infty)$ is some nonexplicitely defined, continuous, nondecreasing function which does not depend on the data of the problem. Obviously, an estimation of the type in (4.3) cannot be obtained from the estimation in (4.4).

We will need the following relaxation result, which is an autonomous, finite dimensional and nonparametrized version of Theorem 6.1 in [8].

THEOREM 4.4. *Assume that* Hypothesis 4.1 *is satisfied and F is compact valued. Then, for any $\epsilon > 0$ and any $z(.) \in AC(I, R^n)$ solution to the problem*

$$z' \in \overline{co}F(z), \quad z(0) = z_0 \in S, \tag{4.4}$$

$$z(t) \in S, \quad \forall t \in [0, T], \tag{4.5}$$

there exists $x_\epsilon(.) \in AC(I, R^n)$ solution to the problem (4.1)–(4.2) *such that for any $t \in I$ one has*

$$||x_\epsilon(t) - z(t)|| \leq \epsilon.$$

We recall that a mapping $V : S \subset R^n \to R$ is called nonincreasing along the function $x : I \to S$ if

$$t_1 < t_2 \implies V(x(t_1)) \geq V(x(t_2)).$$

We are now able to prove the main result of this section.

THEOREM 4.5. *Let $W : S \subset R^n \to R$ be a lower semicontinuous function and let $F : S \to \mathcal{P}(R^n)$ be a compact valued multifunction that satisfies* Hypothesis 4.1 i)–iv). *Then the following statements are equivalent:*
a) *W is nonincreasing along the solutions of* (4.1)–(4.2),
b) $\sup_{u\in F(x)} \underline{D}^+_K W(x; u) \leq 0 \ \forall x \in S,$

c) $\sup_{u\in coF(x)} \underline{D}_K^+ W(x;u) \le 0 \ \forall x\in S$.

Proof. Obviously c) $\Rightarrow$ b). The implication b) $\Rightarrow$ a) is a consequence of Theorem 4.6 in [10]. It remains to prove a) $\Rightarrow$ c).

By contradiction, we assume that there exists $x_0 \in S$ and $u_0 \in coF(x_0)$ such that $\underline{D}_K^+ W(x_0;u_0) > 0$. Since $coF(.)$ is L-lipschitzean (e.g. Proposition 1.3.6 in [1]) we can apply the Filippov existence theorem (e.g. [5], p. 615) and find that there exists $y(.) \in AC(I,R^n)$ solution to the problem

$$y' \in \overline{co}F(y), \quad y(0) = x_0,$$

that, in addition, satisfy $y'(0) = u_0$.

According to Theorem 4.3.8 in [5] (see also Proposition 2.2 in [8]) the condition iv) in Hypothesis 4.1 is equivalent to

$$\overline{co}F(x) \subset K_x S, \quad \forall x\in S.$$

So, we can apply Theorem 4.2 with $p(t) = L\mathrm{d}(y(t),S)$ and deduce that there exists $z(.) \in AC(I,R^n)$ solution to (4.4)–(4.5) such that

$$\int_0^t ||z'(s)-y'(s)||\mathrm{d}s \le \frac{2}{3}\int_0^t [\mathrm{e}^{3L(t-s)}-1]p(s)\mathrm{d}s, \quad \forall t\in I.$$

In particular,

$$||z(t)-y(t)|| \le \frac{2}{3}[\mathrm{e}^{3LT}-1]\int_0^t p(s)\mathrm{d}s, \quad \forall t\in I.$$

Since $p(0)=0$ and

$$\left\|\frac{z(t)-x_0}{t} - \frac{y(t)-x_0}{t}\right\| \le \frac{2}{3}[\mathrm{e}^{3LT}-1]\frac{1}{t}\int_0^t p(s)\mathrm{d}s, \quad \forall t\in I.$$

we infer that $z'(0) = y'(0) = u_0$.

Therefore, there exists $z(.)$ solution to (4.4)–(4.5) such that $z'(0) = u_0$.

On the other hand, one has

$$\liminf_{h\to 0+} \frac{W(z(h))-W(z(0))}{h} = \liminf_{h\to 0+} \frac{W(x_0 + h\frac{z(h)-x_0}{h}) - W(x_0)}{h}$$

$$\ge \liminf_{h\to 0+, u\to u_0} \frac{W(x_0+hu)-W(x_0)}{h} = \underline{D}_K^+ W(x_0;u_0) > 0.$$

Hence, for $\tau > 0$ there exists $\sigma_\tau > 0$ such that

$$W(z(\tau)) > W(z(0)) + \sigma_\tau = W(x_0) + \sigma_\tau.$$

Since $W(.)$ is lower semicontinuous, there exists $\epsilon_\sigma > 0$ such that

$$||x - z(\tau)|| < \epsilon_\sigma \implies W(x) > W(z(\tau)) - \sigma_\tau.$$

We apply now Theorem 4.4 and we find that there exists $x_{\epsilon_\sigma}(.) : [0,\tau] \to R^n$ solution to (4.1)–(4.2) such that

$$||x_{\epsilon_\sigma}(t) - z(t)|| < \epsilon_\sigma, \quad \forall t\in [0,\tau].$$

In particular, $||x_{\epsilon_\sigma}(\tau) - z(\tau)|| < \epsilon_\sigma$.

Thus,

$$W(x_{\epsilon_\sigma}(\tau)) > W(z(\tau)) - \sigma_\tau > -\sigma_\tau + W(x_0) + \sigma_\tau = W(x_0) = W(x_{\epsilon_\sigma}(0)),$$

i.e. $W(x_{\epsilon_\sigma}(\tau)) > W(x_{\epsilon_\sigma}(0))$, which contradicts the hypothesis that W is nonincreasing along the solutions of (4.1)–(4.2). □

REMARK 4.6. If $S = R^n$ then Theorem 4.5 yields Theorem 1.2 in [2].

There are several papers devoted to the study of the monotonicity of solutions of differential inclusions. The first result in the framework of differential inclusions is due to Frankowska ([7]). In Theorem 3.1 in [7] it is proved that if $S = R^n$ and $\sup_{v\in coF(x)} \underline{D}_K^+ W(x;v) \le 0$ then W is nonincreasing along the solutions of (4.1)–(4.2). Mirică (Theorem 4.6 in [10]) proved that $S \subset R^n$ is locally closed and F satisfies a local dissipativity property (instead of lipschitzianity) then condition b) implies a). In [5] a result similar to the one in Theorem 4.5 is proved in the case $S = R^n$ and F is convex valued; this result was extended afterwards in [2] to the nonconvex case.

For a complete discussion and for several refinements and extensions to state constrained differential inclusions of the results in [2] we refer to our paper [4].

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