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UNIVALENCE OF AN INTEGRAL OPERATOR

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Abstract. In this paper we consider the class of univalent functions defined by the condition $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, z \in U$, where $f(z) = z + a_2 z^2 + \ldots$, is an analytic function in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. We present univalence conditions for the operator

$$G_{\alpha,n}(z) = \left((n(\alpha - 1) + 1) \int_0^z g_1^{\alpha-1}(t) \ldots g_n^{\alpha-1}(t)\, dt \right)^{\frac{1}{n(\alpha-1)+1}}.$$

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1. INTRODUCTION

Let U be the unit disc, $U = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $H(U)$ the class of analytic functions on U and consider the set of analytic functions

$$A_n = \left\{ f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots \right\}.$$

If $n = 1$ then $A_1 = A$. Let S be the class of analytic and univalent functions $f(z) = z + a_2 z^2 + \ldots$ in U, which satisfy the condition $f(0) = f'(0) - 1 = 0$.

In their paper [2], Ozaki and Nunokawa proved the following result:

THEOREM 1. *If $f \in A$ satisfies the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \quad z \in U, \tag{1}$$

then f is univalent in U.

LEMMA 1. (The Schwartz Lemma) *Let g be analytic in the unit disc U and $g(0) = 0$. If $|g(z)| \leq 1, \forall z \in U$, then $|g(z)| \leq |z|, \forall z \in U$ and the equality holds if and only if $g(z) = \varepsilon z$, where $|\varepsilon| = 1$.*

THEOREM 2. [3] *Let α be a complex number with $\operatorname{Re}\alpha > 0$, and let $f = z + a_2 z^2 + \ldots$ be an analytic function on U. If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \quad \forall z \in U$$

then for any complex number β with $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$ the function

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t)\, dt \right)^{\frac{1}{\beta}} = z + \ldots$$

is analytic and univalent in U.

THEOREM 3. [4] *Assume that* $g \in A$ *satisfies the condition* (1), *and let* α *be a complex number with* $|\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{3}$. *If* $|g(z)| \leq 1, \forall z \in U$, *then the function*

$$G_\alpha(z) = \left(\alpha \int_0^z g^{\alpha-1}(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}}$$

is of class S.

Hamada [1] showed that Theorem 3 is easily obtained by the Schwarz lemma and gave a generalization as follows:

THEOREM 4. *Let* $M \geq 1$. *Let* $g \in A$ *satisfy the condition* (1) *and* α *be a complex number such that* $\operatorname{Re}\alpha > 0$ *and* $|\alpha - 1| \leq |\alpha| / (2M + 1)$. *If* $|g(z)| \leq M$ *for all* $z \in U$, *then the function*

$$I_\alpha(z) = \left[\alpha \int_0^z g^{\alpha-1}(t)\,\mathrm{d}t\right]^{1/\alpha}$$

is in the class S.

2. MAIN RESULTS

THEOREM 5. *Let* $M \geq 1$, $g_i \in A, \forall i = \overline{1,n}$, $n \in \mathbb{N}^*$, *satisfy the properties*

$$\left|\frac{z^2 g_i'(z)}{g_i^2(z)} - 1\right| < 1, \quad \forall z \in U, \quad \forall i \in \{1, \ldots, n\}$$

and $\alpha \in \mathbb{C}$, *with,* $\operatorname{Re}\{\mathrm{n}(\alpha - 1) + 1\} \geq \operatorname{Re}\alpha$, *and* $|\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{n(2M+1)}$. *If* $|g_i(z)| \leq M$, $\forall z \in U$, $\forall i = \overline{1,n}$, *then the function*

$$G_{\alpha,n}(z) = \left((n(\alpha - 1) + 1) \int_0^z g_1^{\alpha-1}(t) \ldots g_n^{\alpha-1}(t)\,\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}} \tag{2}$$

is univalent.

Proof. From (2) we have:

$$G_{\alpha,n}(z) = \left((n(\alpha - 1) + 1) \int_0^z t^{n(\alpha-1)} \left(\frac{g_1(t)}{t}\right)^{\alpha-1} \ldots \left(\frac{g_n(t)}{t}\right)^{\alpha-1} \mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}}.$$

We consider the function

$$f(z) = \int_0^z \left(\frac{g_1(t)}{t}\right)^{\alpha-1} \ldots \left(\frac{g_n(t)}{t}\right)^{\alpha-1} \mathrm{d}t. \tag{3}$$

Then $f \in A$ and from (3) we obtain

$$f'(z) = \left(\frac{g_1(z)}{z}\right)^{\alpha-1} \cdots \left(\frac{g_n(z)}{z}\right)^{\alpha-1}$$

and

$$f''(z) = (\alpha-1)\left(\frac{g_1(z)}{z}\right)^{\alpha-2} \frac{zg_1'(z)-g_1(z)}{z^2}\left(\frac{g_2(z)}{z}\right)^{\alpha-1} \cdots \left(\frac{g_n(z)}{z}\right)^{\alpha-1}$$
$$+\cdots+\left(\frac{g_1(z)}{z}\right)^{\alpha-1} \cdots \left(\frac{g_{n-1}(z)}{z}\right)^{\alpha-1}(\alpha-1)\left(\frac{g_n(z)}{z}\right)^{\alpha-2} \frac{zg_n'(z)-g_n(z)}{z^2}.$$

Next we calculate the expression $\frac{zf''}{f'}$.

$$\frac{zf''(z)}{f'(z)} = \frac{z\left((\alpha-1)\left(\frac{g_1(z)}{z}\right)^{\alpha-2}\frac{zg_1'(z)-g_1(z)}{z^2}\left(\frac{g_2(z)}{z}\right)^{\alpha-1}\cdots\left(\frac{g_n(z)}{z}\right)^{\alpha-1}\right)}{\left(\frac{g_1(z)}{z}\right)^{\alpha-1}\cdots\left(\frac{g_n(z)}{z}\right)^{\alpha-1}}$$
$$+\cdots+\frac{z\left(\left(\frac{g_1(z)}{z}\right)^{\alpha-1}\cdots\left(\frac{g_{n-1}(z)}{z}\right)^{\alpha-1}(\alpha-1)\left(\frac{g_n(z)}{z}\right)^{\alpha-2}\frac{zg_n'(z)-g_n(z)}{z^2}\right)}{\left(\frac{g_1(z)}{z}\right)^{\alpha-1}\cdots\left(\frac{g_n(z)}{z}\right)^{\alpha-1}}$$
$$=(\alpha-1)\left(\frac{zg_1'(z)-1}{g_1(z)}\right)+\cdots+(\alpha-1)\left(\frac{zg_n'(z)-1}{g_n(z)}\right).$$

The modulus $\left|\frac{zf''}{f'}\right|$ can then be evaluated as

$$\begin{aligned}\left|\frac{zf''(z)}{f'(z)}\right| &= \left|(\alpha-1)\left(\frac{zg_1'(z)-1}{g_1(z)}\right)+\cdots+(\alpha-1)\left(\frac{zg_n'(z)-1}{g_n(z)}\right)\right| \\ &\le \left|(\alpha-1)\left(\frac{zg_1'(z)-1}{g_1(z)}\right)\right|+\cdots+\left|(\alpha-1)\left(\frac{zg_n'(z)-1}{g_n(z)}\right)\right| \\ &= |\alpha-1|\left|\frac{zg_1'(z)-1}{g_1(z)}\right|+\cdots+|\alpha-1|\left|\frac{zg_n'(z)-1}{g_n(z)}\right|.\end{aligned} \tag{4}$$

By multiplying the first and the last terms of (4) by $\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}>0$, we obtain

$$\begin{aligned}\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| &\le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha-1|\left(\left|\frac{zg_1'(z)}{g_1(z)}\right|+1\right) \\ &+\cdots+\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha-1|\left(\left|\frac{zg_n'(z)}{g_n(z)}\right|+1\right) \\ &\le |\alpha-1|\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(\left|\frac{z^2g_1'(z)}{g_1^2(z)}\right|\frac{|g_1(z)|}{|z|}+1\right) \\ &+\cdots+|\alpha-1|\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(\left|\frac{z^2g_n'(z)}{g_n^2(z)}\right|\frac{|g_n(z)|}{|z|}+1\right).\end{aligned} \tag{5}$$

By applying the Schwartz Lemma and using (5), we obtain

$$\begin{aligned}(6)\quad \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| &\le |\alpha-1|\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(\left|\frac{z^2g_1'(z)}{g_1^2(z)}-1\right|M+M+1\right)\\ &+\cdots+|\alpha-1|\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(\left|\frac{z^2g_n'(z)}{g_n^2(z)}-1\right|M+M+1\right).\end{aligned}$$

Since g_i satisfies the condition (1) $\forall i\in\{1,\dots,n\}$, then from (6) we obtain:

$$\begin{aligned}(7)\quad \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| &\le (2M+1)|\alpha-1|\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\\ &+\cdots+(2M+1)|\alpha-1|\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\le\frac{(2M+1)|\alpha-1|}{\operatorname{Re}\alpha}\\ &+\cdots+\frac{(2M+1)|\alpha-1|}{\operatorname{Re}\alpha}=\frac{n(2M+1)|\alpha-1|}{\operatorname{Re}\alpha}.\end{aligned}$$

However $|\alpha-1|\le\frac{\operatorname{Re}\alpha}{n(2M+1)}$ so from (7) we obtain that $\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\le 1$, for all $z\in U$and Theorem 2 implies that $G_{\alpha,n}$ is in the class S. □

COROLLARY 1. *Let* $M\ge 1$, $g\in A$ *satisfy* (1) *and let* $\alpha\in\mathbb{C}$ *be such that* $\operatorname{Re}\{\mathrm{k}(\alpha-1)+1\}\ge \operatorname{Re}\alpha$ *and* $|\alpha-1|\le\frac{\operatorname{Re}\alpha}{k(2M+1)}, k\in\mathbb{N}^*$. *If* $|g(z)|\le M,\forall z\in U$, *then the function*

$$G_\alpha^k(z)=\left((k(\alpha-1)+1)\int_0^z g^{k(\alpha-1)}(t)\,\mathrm{d}t\right)^{\frac{1}{k(\alpha-1)+1}}$$

is univalent.

REMARK 1. Theorem 5 and Corollary 1 are generalizations of Theorem 3.

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