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RIEMANNIAN CONNECTIONS ON SUPERMANIFOLDS: A COORDINATE-FREE APPROACH

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Dedicated to Ştefan Cobzaş, at his 60th anniversary

Abstract. We prove that on any Riemannian supermanifold with a homogeneous (even or odd) metric there exists a single symmetric connection, compatible with the metric and there is provided an explicit global formula for it, without using a coordinate system.
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1. INTRODUCTION

The supermanifolds were introduced in the early seventies and their origin is closely related to theoretical physics. The first differential geometrical approach to supermanifolds ([6]) was also determined by physics, as its aim was to construct a graded counterpart of the geometric prequantization. So far, with the notable exception of the papers of Bejancu ([2], [3]), the Riemannian geometry of supermanifolds have been approached only from a local point of view (see, for instance, [5]). Bejancu ([3]), to our knowledge, was the first to prove the existence and uniqueness of the Levi-Civita connection for an *even* Riemannian supermanifold. However, he "guessed" the formula for the connection by changing in a suitable way the signs in the classical formula. We will show, in this paper, that the formula is, actually, imposed by the properties of symmetry and compatibility with the metric of the connection. Moreover, we will provide a proof for arbitrary homogeneous Riemannian metrics, not necessarily even. It is surprising to see that odd Riemannian supermanifolds appear to have been completely neglected so far in the literature. While the even Riemannian supermanifolds are natural generalizations of the Riemannian manifolds and share many common properties with these ones, the odd Riemannian supermanifolds have no classical analogs, therefore they might be more interesting for the understanding of the geometry of supermanifolds.

2. SUPERMANIFOLDS

The supermanifolds are, loosely speaking, natural generalizations of ordinary manifolds, obtained by using both commuting and anti-commuting coordinate functions. The global definition, however, is a little bit trickier and, in fact, there are several definitions, not completely equivalent between them. We shall use in this paper the definition given by Berezin and Leites in 1975

([4]). We mention that in some places (see, for instance [6]) the Berezin-Leites supermanifolds are called *graded manifolds*. As this is the only version we shall use, there will be no danger of confusion, therefore we shall simply use the term *supermanifold*.

We shall not enter here into any details regarding the definition of supermanifolds, as they will play no role in our work. We will prefer, therefore, to give only the definition, and let the reader consult the classical literature ([7], [8], [1], [5]) for the rest.

DEFINITION 1. Let k be one of the fields $\mathbb{R}$ or $\mathbb{C}$. A *supermanifold* of dimension (m, n) is a pair $\mathcal{M} = (M, \mathcal{A})$, where M is an m-dimensional smooth manifold and $\mathcal{A}$ is a sheaf of graded-commutative $\mathbb{Z}_2$-graded k-algebras (*superalgebras*) on M such that, if $\mathcal{C}^\infty_M$ is the sheaf of smooth functions on M, then

(i) there is an exact sequence of sheaves

$$0 \to \mathcal{I} \to \mathcal{A} \xrightarrow{\pi} \mathcal{C}^\infty_M \to 0,$$

where, for any open set $U \subset M$, $\pi_U : \mathcal{A}(U) \to C^\infty(U)$ is a surjective morphism of graded algebras, while $\mathcal{I} = \mathcal{A}_1 + \mathcal{A}_1^2$ is the sheaf of nilpotents of the sheaf $\mathcal{A}$.

(ii) $\mathcal{I}/\mathcal{I}^2$ is a locally free of rank n over $\mathcal{C}^\infty_M$ and $\mathcal{A}$ is locally-isomorphic, as a sheaf of graded-commutative superalgebras, to the sheaf of exterior algebras $\Lambda_{\mathcal{C}^\infty_M}(\mathcal{I}/\mathcal{I}^2)$.

REMARK 1. The second condition from the definition of the supermanifold actually means that, locally, $\mathcal{A}$ can be written as

$$\mathcal{A}(U) \simeq C^\infty(U) \otimes \Lambda_n,$$

where $U \subset M$ is an open set, while Λ_n is a Grassmann algebra with n generators.

On an ordinary manifold M, a basic role is played by the algebra of smooth, real-valued functions on M, $C^\infty(M)$. On a supermanifold $(M, \mathcal{A})$, on the other hand, a similar role is played by the superalgebra of global sections of the sheaf $\mathcal{A}$, $\mathcal{A}(M)$. As we mentioned already, this is a $\mathbb{Z}_2$-graded algebra, which means that it can be decomposed (as a module) into a direct sum

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1.$$

The elements of $\mathcal{A}_i$, $i = 0, 1$ are called *homogeneous* and their degree is also called the *parity*. More specifically, the elements of $\mathcal{A}_0$ are called *even*, while those of $\mathcal{A}_1$ are called *odd*. The parity of a homogeneous element $f \in \mathcal{A}$ is denoted by $|f|$.

Using the superalgebra of global sections on a supermanifold, we can define objects similar to those associated to an ordinary supermanifold.

Definition 2. Let $\mathcal{M} = (M, \mathcal{A})$ be a supermanifold. A (graded) *vector field* on $\mathcal{M}$ is a graded derivation of $\mathcal{A}(M)$, i.e. a linear map $X : \mathcal{A} \to \mathcal{A}$ which satisfy the graded Leibniz rule:

$$X(f \cdot g) = X(f) \cdot g + (-1)^{|X||f|} f \cdot X(g),$$

for any $f, g \in \mathcal{A}$. Here X is even if it preserves the parity and is odd if it reverse it.

It is easy to see that the set of all vector fields on a supermanifold has a natural structure of $\mathbb{Z}_2$-graded $\mathcal{A}(M)$-module that we will denote, as in the case of manifolds, by $\mathcal{X}(M)$.

A very important operation with vector fields on a supermanifold is the *graded Lie brackets*:

Definition 3. If $X, Y \in \mathcal{X}(M)$, their graded Lie bracket is defined by

$$[X, Y](f) = X(Y(f)) - (-1)^{|X||Y|} Y(X(f)),$$

for any $f \in \mathcal{A}(M)$.

It is easy to check that for each pair of vector fields, their graded Lie bracket (also called the *super-commutator*) is, indeed, a vector field. It is, equally, easy to check that, unlike the ordinary Lie bracket, which is anti-commutative, the graded Lie bracket is *graded*-anti-commutative, i.e. for each pair of vector fields X, Y we have

$$[X, Y] = -(-1)^{|X||Y|}[Y, X].$$

3. CONNECTIONS ON SUPERMANIFOLDS

Definition 4. Let $\mathcal{M} \equiv (M, \mathcal{A})$ be a supermanifold and $\mathcal{X}(M)$ – the supermodule of graded vector fields on $\mathcal{M}$. A *linear connection* on $\mathcal{M}$ is an *even* map

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M), \quad (X, Y) \to \nabla_X Y$$

such that the following properties are fulfilled:

(1) ∇ is linear in the first argument: for any $f_1, f_2 \in \mathcal{A}(M)$ and X_1, X_2, $Y \in \mathcal{X}(M)$, we have

$$\nabla_{f_1 X_1 + f_2 X_2} = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y;$$

(2) ∇ is additive in the second argument: for any $X, Y_1, Y_2 \in \mathcal{X}(M)$ we have

$$\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2;$$

(3) for any $X \in \nabla$, ∇_X verifies the graded Leibniz identity: for any $f \in \mathcal{A}(M)$ and $Y \in \nabla$, X and f homogeneous, we have

$$\nabla_X (f \cdot Y) = X(f) \cdot Y + (-1)^{|X||f|} f \cdot \nabla_X Y.$$

4. RIEMANNIAN METRICS AND CONNECTIONS

DEFINITION 5. A *Riemannian metric* on a supermanifold is a bilinear form $g : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, which is, also, subject to the further restrictions:

(i) symmetry: for any homogeneous vector fields $X, Y \in \mathcal{X}(M)$, we have

$$g(Y, X) = (-1)^{|X||Y|} g(X, Y);$$

(ii) non-degeneracy: for any $X \in \mathcal{X}(M)$, the map $Y \to g(X, Y)$ is a linear isomorphism.

The existence of Riemannian metrics on supermanifold is a direct consequence of the existence of a graded partition of unity and the proof is analogous to the one from the non-graded case. Actually, it can be seen immediately that the set of all Riemannian metrics on a given supermanifold carries a natural structure of $\mathbb{Z}_2$-graded space. Let us assume, hereafter, that g is a fixed, homogeneous metric on the supermanifold $\mathcal{M} = (M, \mathcal{A})$.

We have to notice that the $\mathcal{A}(M)$-module $\mathcal{X}(M)$ is, actually, a $\mathcal{A}(M)$-bimodule and, due to the connection between the left and the right structures on it, we have, for the metric g, the following identities, for any homogeneous $f \in \mathcal{A}(M)$ and $X, Y \in \mathcal{X}(M)$:

(i) $g(f \cdot X, Y) = (-1)^{|f||g|} g \cdot g(X, Y)$;
(ii) $g(X \cdot f, Y) = g(X, f \cdot Y)$;
(iii) $g(X, Y \cdot f) = g(X, Y) \cdot f$.

DEFINITION 6. Let $\mathcal{M}$ be a supermanifold and g a Riemannian metric on $\mathcal{M}$. A linear connection ∇ on $\mathcal{M}$ is called

(i) symmetrical if, for any $X, Y \in \mathcal{X}(M)$ we have

$$\nabla_X Y = (-1)^{|X||Y|} \nabla_Y X + [X, Y];$$

(ii) compatible with the metric if, for any $X, Y, Z \in \mathcal{X}(M)$ we have

$$g(\nabla_X Y, Z) = (-1)^{|X||g|} X g(Y, Z) - (-1)^{|X||Y|} g(Y, \nabla_X Z).$$

THEOREM 1. *On any homogeneous Riemannian supermanifold there exist a unique symmetrical connection, compatible with the metric. This connection is called, as in the classical Riemannian geometry,* the Levi-Civita connection *associated with the given Riemannian metric.*

Proof. We have

$$\begin{aligned}
&g(\nabla_X Y, Z) = (-1)^{|X||g|} X g(Y, Z) - (-1)^{|X||Y|} g(Y, \nabla_X Z) \\
&= (-1)^{|X||g|} X g(Y, Z) - (-1)^{|X||Y|} g(Y, [X, Z]) - (-1)^{|X||Y| + |X||Z|} g(Y, \nabla_Z X) \\
&= (-1)^{|X||g|} X g(Y, Z) - (-1)^{|X||Y|} g(Y, [X, Z]) \\
&- (-1)^{|X||Y| + |X||Z|} \left[(-1)^{|Z||Y| + |g|)} Z g(Y, X) - (-1)^{|Z||Y|} g(\nabla_Z Y, X) \right]
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{|X||g|}Xg(Y,Z) - (-1)^{|X||Y|}g(Y,[X,Z]) \\
&- (-1)^{|X||Y|+|Y||Z|+|Z||X|+|Z||g|}Zg(Y,X) + (-1)^{|X||Y|+|Y||Z|+|X||Z|}g(\nabla_Z Y,X) \\
&= (-1)^{|X||g|}Xg(Y,Z) - (-1)^{|X||Y|}g(Y,[X,Z]) \\
&- (-1)^{|X||Y|+|Y||Z|+|Z||X|+|Z||g|}Zg(Y,X) + (-1)^{|X||Y|+|Y||Z|+|X||Z|}g([Z,Y],X) \\
&+ (-1)^{|X||Y|+|X||Z|}g(\nabla_Y Z,X) = (-1)^{|X||g|}Xg(Y,Z) - (-1)^{|X||Y|}g(Y,[X,Z]) \\
&- (-1)^{|X||Y|+|Y||Z|+|Z||X|+|Z||g|}Zg(Y,X) + (-1)^{|X||Y|+|Y||Z|+|X||Z|}g([Z,Y],X) \\
&+ (-1)^{|X||Y|+|X||Z|}\Big[(-1)^{|Y||g|}Yg(Z,X) - (-1)^{|Y||Z|}g(Z,\nabla_Y X)\Big] \\
&= (-1)^{|X||g|}Xg(Y,Z) - (-1)^{|X||Y|}g(Y,[X,Z]) \\
&- (-1)^{|X||Y|+|Y||Z|+|Z||X|+|Z||g|}Zg(Y,X) + (-1)^{|X||Y|+|Y||Z|+|X||Z|}g([Z,Y],X) \\
&+ (-1)^{|X||Y|+|X||Z|+|Y||g|}Yg(Z,X) - (-1)^{|X||Y|+|Y||Z|+|X||Z|}g(Z,\nabla_Y X) \\
&= (-1)^{|X||g|}Xg(Y,Z) - (-1)^{|X||Y|}g(Y,[X,Z]) \\
&- (-1)^{|X||Y|+|Y||Z|+|Z||X|+|Z||g|}Zg(Y,X) + (-1)^{|X||Y|+|Y||Z|+|X||Z|}g([Z,Y],X) \\
&- (-1)^{|X||Y|+|X||Z|+|Y||Z|}g(Z,[Y,X]) - (-1)^{|X||Z|+|Y||Z|}g(Z,\nabla_X Y) \\
&+ (-1)^{|X||Y|+|X||Z|+|Y||g|}Yg(Z,X) = (-1)^{|X||g|}Xg(Y,Z) \\
&+ (-1)^{|X|(|Y|+|Z|)+|Y||g|}Yg(Z,X) - (-1)^{|Z|(|X|+|Y|)+|Z||g|}Zg(X,Y) \\
&+ g([X,Y],Z) - (-1)^{|X|(|Y|+|Z|)}g([Y,Z],X) + (-1)^{|Z|(|X|+|Y|)}g([Z,X],Y) \\
&- g(\nabla_X Y,Z),
\end{aligned}
$$

whence

$$
\begin{aligned}
&2\cdot g(\nabla_X Y,Z) = (-1)^{|X||g|}Xg(Y,Z) + (-1)^{|X|(|Y|+|Z|)+|Y||g|}Yg(Z,X) \\
&- (-1)^{|Z|(|X|+|Y|)+|Z||g|}Zg(X,Y) + g([X,Y],Z) \\
&- (-1)^{|X|(|Y|+|Z|)}g([Y,Z],X) + (-1)^{|Z|(|X|+|Y|)}g([Z,X],Y).
\end{aligned} \tag{1}
$$

In particular, for *even* Riemannian metrics we regain the formula (3.10) of Bejancu's paper ([3]), i.e.

$$
\begin{aligned}
2\cdot g(\nabla_X Y,Z) &= Xg(Y,Z) + (-1)^{|X|(|Y|+|Z|)}Yg(Z,X) \\
&- (-1)^{|Z|(|X|+|Y|)}Zg(X,Y) + g([X,Y],Z) \\
&- (-1)^{|X|(|Y|+|Z|)}g([Y,Z],X) + (-1)^{|Z|(|X|+|Y|)}g([Z,X],Y).
\end{aligned}
$$

We mention, however, that Bejancu *postulated* this identity, while we obtained it from the properties of the connection.

As the metric is nondegenerate, the formula (1) uniquely defines the connection, provided it exists.

The existence proof is parallel to the similar proof from classical Riemannian geometry. Namely, we take (1) as *the definition* of the Riemannian connection and then we prove that this formula gives, indeed, a symmetric connection which is compatible with the Riemannian metric.

First of all, let us prove that the operator ∇ has the right parity, namely zero. Clearly, this simply means that $|\nabla_X Y| = |X| + |Y|$. But, in (1), the left hand size has the parity $|g|+|\nabla_X Y|+|Z|$, while all the terms of the right hand side have the parity $|g| + |X| + |Y| + |Z|$, which proves our claim. Linearity in X and additivity in Y is no problem. To prove that ∇ is a connection we still have to prove the graded Leibniz rule, in other words, we have to prove the identity

$$\nabla_X(f \cdot Y) = X(f) \cdot Y + (-1)^{|X|\cdot|f|} f \cdot \nabla_X Y. \tag{2}$$

We start from the relation (1) and we get

$$\begin{aligned}
&2 \cdot g(\nabla_X(f \cdot Y), Z) = (-1)^{|X||g|} X g(f \cdot Y, Z)\\
&+ (-1)^{|X|(|f|+|Y|+|Z|)+(|f|+|Y|)|g|}(f \cdot Y) g(Z, X)\\
&- (-1)^{|Z|(|X|+|f|+|Y|)+|Z||g|} Z g(X, f \cdot Y) + g([X, f \cdot Y], Z)\\
&- (-1)^{|X|(|Y|+|Z|)} g([f \cdot Y, Z], X)\\
&+ (-1)^{|Z|(|X|+|f|+|Y|)} g([Z, X], f \cdot Y).
\end{aligned} \tag{3}$$

We shall take out separately the six terms of the right hand side of the formula (3) and we shall work them out independently. We have, thus,

$$\begin{aligned}
&(-1)^{|X||g|} X g(f \cdot Y, Z) = (-1)^{|X||g|} \left[X\left((-1)^{|f||g|} f \cdot g(Y, Z)\right)\right]\\
&= (-1)^{|g|(|X|+|f|)} X(f) \cdot g(Y, Z)\\
&+ (-1)^{|g|(|X|+|f|)+|X||f|} f \cdot X g(Y, Z) = g(X(f) \cdot Y, Z)\\
&+ (-1)^{|g|(|X|+|f|)+|X||f|} f \cdot X g(Y, Z),
\end{aligned} \tag{4}$$

$$\begin{aligned}
&(-1)^{|X|(|f|+|Y|+|Z|)+(|f|+|Y|)|g|}(f \cdot Y) g(Z, X)\\
&\quad = (-1)^{|X|(|f|+|Y|+|Z|)+(|f|+|Y|)|g|} f \cdot Y g(Z, X),
\end{aligned} \tag{5}$$

$$\begin{aligned}
&(-1)^{|Z|(|X|+|f|+|Y|)+|Z||g|} Z g(X, f \cdot Y)\\
&= (-1)^{|Z|(|X|+|f|+|Y|)+|Z||g|} Z g(X \cdot f, Y)\\
&= (-1)^{|Z|(|X|+|f|+|Y|)+|Z||g|+|X||f|} Z g(f \cdot X, Y)\\
&= (-1)^{|Z|(|X|+|f|+|Y|+|g|)+|f|(|X|+|g|)} Z\left[f \cdot g(X, Y)\right]\\
&= (-1)^{|Z|(|X|+|f|+|Y|+|g|)+|f|(|X|+|g|)} Z(f) \cdot g(X, Y)\\
&+ (-1)^{|Z|(|X|+|Y|+|g|)+|f|(|X|+|g|)} f \cdot Z g(X, Y)\\
&= (-1)^{|Z|(|X|+|f|+|Y|)+|f||X|)} g(Z(f) \cdot X, Y)\\
&+ (-1)^{|Z|(|X|+|Y|+|g|)+|f|(|X|+|g|)} f \cdot Z g(X, Y),
\end{aligned} \tag{6}$$

$$
\begin{aligned}
&g([X, f\cdot Y], Z) = g\left(X(f)\cdot Y + (-1)^{|X||f|} f\cdot [X,Y], Z\right)\\
&= g(X(f)\cdot Y, Z) + (-1)^{|f|(|X|+|g|)} f\cdot g([X,Y],Z),
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&(-1)^{|X|(|Y|+|Z|)} g([f\cdot Y, Z], X)\\
&= (-1)^{|X|(|Y|+|Z|)} g\left(-(-1)^{|Z|(|f|+|Y|)} Z(f)\cdot Y + f\cdot [Y,Z], X\right)\\
&= -(-1)^{|X|(|Y|+|Z|)+|Z|(|f|+|Y|)} g(Z(f)\cdot Y, X)\\
&+ (-1)^{|X|(|Y|+|Z|)+|g||f|} f\cdot g([Y,Z],X),
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
&(-1)^{|Z|(|X|+|f|+|Y|)} g([Z,X], f\cdot Y)\\
&= (-1)^{|Z|(|X|+|f|+|Y|)} g([Z,X]\cdot f, Y)\\
&= (-1)^{|Z|(|X|+|Y|)+|f||X|} g(f\cdot [Z,X], Y)\\
&= (-1)^{|Z|(|X|+|Y|)+|f|(|X|+|g|)} f\cdot g([Z,X],Y).
\end{aligned} \tag{9}
$$

Now, substituting the relations (4)–(9) into (3), we get

$$
\begin{aligned}
&2\cdot g(\nabla_X(f\cdot Y), Z) = 2\cdot g(X(f)\cdot Y, Z)\\
&+ (-1)^{|X||f|+|g||f|} f\cdot \Big[(-1)^{|X||g|} X g(Y,Z)\\
&+ (-1)^{|X|(|Y|+|Z|)+|Y||g|} Y g(Z,X)\\
&- (-1)^{|Z|(|X|+|Y|)+|Z||g|} Z g(X,Y)\\
&+ g([X,Y],Z) - (-1)^{|X|(|Y|+|Z|)} g([Y,Z],X)\\
&+(-1)^{|Z|(|X|+|Y|)} g([Z,X],Y)\Big]\\
&= 2\cdot g(X(f)\cdot Y, Z) + (-1)^{|X||f|+|g||f|} 2\cdot f\cdot g(\nabla_X Y, Z)\\
&= 2\cdot g(X(f)\cdot Y, Z) + (-1)^{|X||f|} 2\cdot g(f\cdot \nabla_X Y, Z)\\
&= 2\cdot g(X(f)\cdot Y + (-1)^{|X||f|} f\cdot \nabla_X Y, Z),
\end{aligned}
$$

hence

$$
\nabla_X(f\cdot Y) = X(f)\cdot Y + (-1)^{|X||f|} f\cdot \nabla_X Y. \qquad \square
$$

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