MATHEMATICA, Tome 47 (70), N^o 1, 2005, pp. 3–18

ON A CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH COMPLEX ORDER DEFINED BY SALAGEAN OPERATOR

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Abstract. We introduce a class, namely $R_\alpha^n(b,\beta)(b \neq 0$, complex, $0 < \beta \leq 1$, $n \in N_0 = \{0,1,2,\dots\}$ and $0 \leq \alpha < 1)$ of analytic functions defined by using Hadamard product $(D^n f * S_\alpha)(z)$ of the differential operator $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$ and $S_\alpha(z) = \dfrac{z}{(1-z)^{2(1-\alpha)}}$ and satisfying the condition

$$\left| \frac{(D^n f * S_\alpha)'(z) - 1}{2\beta\left[(D^n f * S_\alpha)'(z) - 1 + b\right] - \left[(D^n f * S_\alpha)'(z) - 1\right]} \right| < 1, \quad z \in U.$$

In this paper we determine a sufficient condition, coefficient estimates, maximization of $\left|a_3 - \mu a_2^2\right|$ over the class $R_\alpha^n(b,\beta)$, distortion theorem and an argument theorem for the class $R_\alpha^n(b,\beta)$. Further we prove that some of the subclasses of $R_\alpha^n(b,\beta)$ are closed under convolution.

MSC 2000. 30C45.
Key words. Analytic, distortion, convolution.

1. INTRODUCTION

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Also let S denote the subclass of A consisting of analytic and univalent functions $f(z)$ in U. We use Ω to denote the class of bounded analytic functions $\omega(z)$ in U which satisfy the conditions $\omega(0) = 0$ and $|\omega(z)| \leq |z|$ for $z \in U$. If the function $f(z) \in A$ satisfies the condition

$$\operatorname{Re}\left\{f'(z)\right\} > 0, \quad z \in U, \tag{1.2}$$

then it is well known that $f(z)$ is univalent in U. We denote the class of such functions by R. This class was introduced and studied by MacGregor [13].

Let R_α denotes the class of functions $f(z) \in A$ that satisfy the condition

$$\operatorname{Re}\left\{f'(z)\right\} > \alpha, \quad 0 \leq \alpha < 1, \quad z \in U. \tag{1.3}$$

The class R_α was studied by Ezrohi [5]. Clearly $R_0 \equiv R$.

A function $f(z) \in S$ is said to be starlike of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in U, \tag{1.4}$$

for some $\alpha\,(0 \leq \alpha < 1)$. We denote the class of all starlike functions of order α by $S^*(\alpha)$.

Now, the function

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \tag{1.5}$$

is the well-known extremal function for the class $S^*(\alpha)$ (see [20]).

Setting

$$C(\alpha, k) = \frac{\prod_{p=2}^{k}(p-2\alpha)}{(k-1)!} \quad (k \geq 2), \tag{1.6}$$

$S_\alpha(z)$ can be written in the form

$$S_\alpha(z) = z + \sum_{k=2}^{\infty} C(\alpha, k)\, z^k. \tag{1.7}$$

Then we note that $C(\alpha, k)$ is a decreasing function in α and satisfies

$$\lim_{k\to\infty} C(\alpha, k) = \begin{cases} \infty\,, & \alpha < \frac{1}{2} \\ 1\,, & \alpha = \frac{1}{2} \\ 0\,, & \alpha > \frac{1}{2} \end{cases} .$$

Let $(f * g)(z)$ be the convolution or Hadamard product of two functions $f(z)$ and $g(z)$, that is, $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.8}$$

then

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.9}$$

For a function $f(z) \in S$, we define

$$D^0 f(z) = f(z)\,, \tag{1.10}$$

$$D^1 f(z) = Df(z) = zf'(z)\,, \tag{1.11}$$

and

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \qquad (n \in N = \{1, 2, \dots\})\,. \tag{1.12}$$

It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad n \in N_0 = N \cup \{0\}\,. \tag{1.13}$$

The differential operator D^n was introduced by Salagean [21].

In this paper, we introduce the class $R_{\alpha}^{n}(b,\beta)$ of functions $f(z) \in A$, defined as follows:

DEFINITION 1. *Let $f(z) \in A$; and let $b \neq 0$, complex, $0 < \beta \leq 1$, $n \in N_0$ and $0 \leq \alpha < 1$. Then $f(z)$ is said to be in the class $R_{\alpha}^{n}(b,\beta)$ if it satisfies the condition*

$$\left| \frac{(D^n f * S_\alpha)'(z) - 1}{2\beta\left[(D^n f * S_\alpha)'(z) - 1 + b\right] - \left[(D^n f * S_\alpha)'(z) - 1\right]} \right| < 1, \tag{1.14}$$

for all $z \in U$.

We note that $R_{\frac{1}{2}}^{0}(1,1) \equiv R$, $R_{\frac{1}{2}}^{0}(1-\rho,1) \equiv R_\rho$ $(0 \leq \rho < 1)$ and

$$R_{\frac{1}{2}}^{0}(b,\beta) = \left\{ f(z) \in A : \left| \frac{f'(z) - 1}{2\beta(f'(z) - 1 + b) - (f'(z) - 1)} \right| < 1,\ z \in U \right\}.$$

By taking different values of b, β, n and α, the class $R_{\alpha}^{n}(b,\beta)$ reduces to various well known subclasses of R; for example,

(1) $R_{\frac{1}{2}}^{0}(b,1) = R(b)$ (Abdul Halim [1]);

(2) $R_{\frac{1}{2}}^{0}\left((1-\rho)\,\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, \beta\right) = R^{\lambda}(\rho,\beta)$ $\left(|\lambda| < \frac{\pi}{2},\ 0 \leq \rho < 1,\ 0 < \beta \leq 1\right)$ (Ahuja [2]);

(3) $R_{\frac{1}{2}}^{0}\left(\frac{2\beta(1-\rho)\,\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda}{1+\beta}, \frac{1+\beta}{2}\right) = R_{\rho,\beta}^{\lambda}$ $\left(|\lambda| < \frac{\pi}{2},\ 0 \leq \rho < 1,\ 0 < \beta \leq 1\right)$ (Makówka [14] and Gopalakrishna and Umarani [9]);

(4) $R_{\frac{1}{2}}^{0}\left(\frac{2\gamma}{1+\gamma}, \frac{1+\gamma}{2}\right) = R(\gamma)$ $(0 < \gamma \leq 1)$(Padmanabhan [19] and Caplinger and Causey [3]);

(5) $R_{\frac{1}{2}}^{0}\left(1, \frac{1}{2}\right) = R^{*}$ (MacGregor [12]);

(6) $R_{\frac{1}{2}}^{0}\left(\sigma, \frac{1}{2}\right) = R^{*}(\sigma)$ $(0 < \sigma \leq 1)$ (Goel [6]);

(7) $R_{\frac{1}{2}}^{0}\left(1, \frac{2\delta-1}{2\delta}\right) = S(\delta)$ $\left(\delta > \frac{1}{2}\right)$ (Goel [7, 8]);

(8) $R_{\frac{1}{2}}^{0}\left(1-a-d, \frac{1-a+d}{2d}\right) = S(a,d)$ $(a+d \geq 1,\ d \leq a \leq d+1)$ (Chen [4] and Owa [18]);

(9) $R_{\frac{1}{2}}^{0}\left(\sigma\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, \frac{1}{2}\right) = \left(R_1^{\lambda}\right)^{\sigma}$ $\left(|\lambda| < \frac{\pi}{2},\ 0 < \sigma \leq 1\right)$ (Mogra [16]);

(10) $R_{\frac{1}{2}}^{0}\left(\frac{2\gamma}{1+\gamma}\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, \frac{1+\gamma}{2}\right) = \left(R_1^{\lambda}\right)_{\gamma}$ $\left(|\lambda| < \frac{\pi}{2},\ 0 < \gamma \leq 1\right)$ (Mogra [16]);

(11) $R_{\frac{1}{2}}^{0}\left(\frac{2(1-\rho)\beta}{1+\beta}, \frac{1+\beta}{2}\right) = R(\rho,\beta)$ $(0 \leq \rho < 1,\ 0 < \beta \leq 1)$ (Juneja and Mogra [10]);

(12) $R_{\frac{1}{2}}^{0}(1-\rho,\beta) = R_1(\rho,\beta)$ $(0 \leq \rho < 1,\ 0 < \beta \leq 1)$ (Mogra [15]);

(13) $R^0_{\frac{1}{2}}\left((1-\rho)\,\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, 1\right) = R^\lambda_\rho$ $(|\lambda| < \frac{\pi}{2},\ 0 \le \rho < 1)$ (Ahuja [2]);

(14) $R^0_{\frac{1}{2}}\left(\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, \dfrac{2-\cos\lambda}{2}\right) = R^{*\lambda}$ (Ahuja [2]);

(15) $R^0_{\frac{1}{2}}\left(\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, \dfrac{2\delta-1}{2\delta}\right) = R^{*\lambda}_\delta$ $(|\lambda| < \frac{\pi}{2}, \delta > \frac{1}{2})$ (Ahuja [2]);

(16) $R^0_{\frac{1}{2}}\left(\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, 1-\rho\right) = R^{*\lambda}(\rho)$ $(|\lambda| < \frac{\pi}{2}, 0 \le \rho < 1)$ (Ahuja [2]).

We further, observe that by special choices of b and β our class $R^n_\alpha(b,\beta)$ give rise to the following new subclasses of R:

(1) $R^n_\alpha(b,1) = R^n_\alpha(b)$

$$= \left\{ f(z) \in A : \operatorname{Re}\left\{ 1 + \frac{1}{b}\left((D^n f * S_\alpha)'(z) - 1 \right) \right\} > 0,\ z \in U \right\};$$

(2) $R^n_\alpha\left(1-\rho, \frac{1}{2}\right) = R^{*n}_\alpha(\rho)$

$$= \left\{ f(z) \in A : \left| (D^n f * S_\alpha)'(z) - 1 \right| < 1-\rho,\ 0 \le \rho < 1,\ z \in U \right\};$$

(3) $R^n_\alpha\left(b, \dfrac{2\delta-1}{2\delta}\right) = R^n_\alpha(b,\delta)$

$$= \left\{ f(z) \in A : \left| \frac{b-1+(D^n f * S_\alpha)'(z)}{b} - \delta \right| < \delta, \quad \delta > \frac{1}{2}, \quad z \in U \right\};$$

(4) $R^n_\alpha\left((1-\rho)\,\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, \dfrac{2\delta-1}{2\delta}\right) = R^{*n,\lambda}_{\alpha,\delta}(\rho)$

$$= \left\{ f(z) \in A : \left| \frac{\mathrm{e}^{\mathrm{i}\lambda}(D^n f * S_\alpha)'(z) - \rho\cos\lambda - \mathrm{i}\sin\lambda}{(1-\rho)\cos\lambda} - \delta \right| < \delta, \right.$$
$$\left. |\lambda| < \frac{\pi}{2},\ 0 \le \rho < 1,\ \delta > \frac{1}{2},\ z \in U \right\};$$

(5) $R^n_\alpha\left((1-\rho)\,\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, 1-\xi\right) = R^{*n,\lambda}_\alpha(\xi,\rho)$

$$= \left\{ f(z) \in A : \left| \frac{\mathrm{e}^{\mathrm{i}\lambda}(D^n f * S_\alpha)'(z) - \rho\cos\lambda - \mathrm{i}\sin\lambda}{(1-\rho)\cos\lambda} - \frac{1}{2\xi} \right| < \frac{1}{2\xi}, \right.$$
$$\left. |\lambda| < \frac{\pi}{2},\ 0 \le \rho < 1,\ 0 \le \xi < 1,\ z \in U \right\};$$

(6) $R^n_\alpha\left((1-\rho)\,\sigma\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, \frac{1}{2}\right) = \left(R^{n,\lambda}_{\alpha,1}(\rho)\right)^\sigma$

$$= \left\{ f(z) \in A : \left| \frac{\mathrm{e}^{\mathrm{i}\lambda}(D^n f * S_\alpha)'(z) - \rho\cos\lambda - \mathrm{i}\sin\lambda}{(1-\rho)\cos\lambda} - 1 \right| < \sigma, \right.$$
$$\left. |\lambda| < \frac{\pi}{2},\ 0 \le \rho < 1,\ 0 < \sigma \le 1,\ z \in U \right\};$$

$$(7)\ R_\alpha^n\left([(1-a)+d]\,\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, \frac{(1-a)+d}{2d}\right) = R_\alpha^{n,\lambda}(a,d)$$

$$= \left\{f(z)\in A : \left|\frac{\mathrm{e}^{\mathrm{i}\lambda}\left(D^n f * S_\alpha\right)'(z) - \mathrm{i}\sin\lambda}{\cos\lambda} - a\right| < d,\right.$$

$$\left.|\lambda| < \frac{\pi}{2},\ a+d>1,\ d\le a\le d+1,\ z\in U\right\};$$

$$(8)\ R_\alpha^n\left([(1-m)-M](1-\rho)\,\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda, \frac{(1-m)+M}{2M}\right) = R_{\alpha,m,M}^{n,\lambda}(\rho)$$

$$= \left\{f(z)\in A : \left|\frac{\mathrm{e}^{\mathrm{i}\lambda}\left(Df * S_\alpha\right)'(z) - \rho\cos\lambda - \mathrm{i}\sin\lambda}{(1-\rho)\cos\lambda} - m\right| < M,\right.$$

$$\left.|\lambda| < \frac{\pi}{2},\ 0\le\rho<1,\ |m-1|<M\le m,\ m>\frac{1}{2},\quad z\in U\right\}.$$

2. A SUFFICIENT CONDITION

THEOREM 1. *The function $f(z)$ defined by (1.1) is in the class $R_\alpha^n(b,\beta)$, if for $b\neq 0$, complex, $n\in N_0$, and $0\le\alpha<1$,*

$$\sum_{k=2}^{\infty} k^{n+1}C(\alpha,k)\,|a_k| \le \frac{\beta\,|b|}{1-\beta}, \tag{2.1}$$

whenever $\beta\in(0,\frac{1}{2}]$, and

$$\sum_{k=2}^{\infty} k^{n+1}C(\alpha,k)\,|a_k| \le |b|, \tag{2.2}$$

whenever $\beta\in[\frac{1}{2},1]$, holds.

Proof. Let $|z|=r<1$, and suppose $0<\beta\le\frac{1}{2}$. Then

$$\left|(D^n f * S_\alpha)'(z) - 1\right| - \left|2\beta\left[(D^n f * S_\alpha)'(z) - 1 + b\right]\right.$$

$$-\ \left.\left[(D^n f * S_\alpha)'(z) - 1\right]\right| = \left|\sum_{k=2}^{\infty} k^{n+1}C(\alpha,k)\,a_k z^{k-1}\right|$$

$$-\ \left|2\beta b - (1-2\beta)\sum_{k=2}^{\infty} k^{n+1}C(\alpha,k)\,a_k z^{k-1}\right| \le \sum_{k=2}^{\infty} k^{n+1}C(\alpha,k)\,|a_k|\,r^{k-1}$$

$$-\ \left\{2\beta\,|b| - (1-2\beta)\sum_{k=2}^{\infty} k^{n+1}C(\alpha,k)\,|a_k|\,r^{k-1}\right\}$$

$$\le\ 2\left[(1-\beta)\sum_{k=2}^{\infty} k^{n+1}C(\alpha,k)\,|a_k| - \beta\,|b|\right].$$

The last quantity is nonpositive by (2.1), so that $f(z) \in R_\alpha^n(b,\beta)$. Next, we assume that (2.2) holds for $\frac{1}{2} \leq \beta \leq 1$. Then

$$\begin{aligned} & \left|(D^n f * S_\alpha)'(z) - 1\right| - \left|2\beta\left[(D^n f * S_\alpha)'(z) - 1 + b\right]\right. \\ - & \left.\left[(D^n f * S_\alpha)'(z) - 1\right]\right| = \left|\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k) a_k z^{k-1}\right| \\ - & \left|2\beta b - (1-2\beta)\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k) a_k z^{k-1}\right| \\ \leq & \; 2\beta\left[\sum_{k=2}^{\infty} k^{n+1} C(\alpha,k)|a_k| - |b|\right] \leq 0. \end{aligned}$$

This proves that $f(z) \in R_\alpha^n(b,\beta)$, hence the theorem. □

We note that

$$f(z) = z + \frac{\beta b}{(1-\beta) k^{n+1} C(\alpha,k)} z^k \quad (k \geq 2),$$

is an extremal function with respect to the first part of the theorem and

$$f(z) = z + \frac{b}{k^{n+1} C(\alpha,k)} z^k \quad (k \geq 2),$$

is an extremal function with respect to the second part of the theorem, since

$$\left|\frac{(D^n f * S_\alpha)'(z) - 1}{2\beta\left[(D^n f * S_\alpha)'(z) - 1 + b\right] - \left[(D^n f * S_\alpha)'(z) - 1\right]}\right| = 1$$

for $z = 1$, $b \neq 0$, complex, $0 < \beta \leq 1$, $n \in N_0$, $0 \leq \alpha < 1$, and $k \geq 2$.

We also observe that the converse of the above theorem may not be true. For example, consider the function $(D^n f * S_\alpha)'(z)$ defined by

$$(D^n f * S_\alpha)'(z) = \frac{1 - (2\beta - 1 - 2\beta b) z}{1 - (2\beta - 1) z}.$$

It is easily seen that $f(z) \in R_\alpha^n(b,\beta)$ but

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{k^{n+1} C(\alpha,k)(1-\beta)}{\beta|b|} |a_k| \\ = & \sum_{k=2}^{\infty} \frac{k^{n+1} C(\alpha,k)(1-\beta)}{\beta|b|} \cdot \frac{2\beta|b|(2\beta-1)^{k-2}}{k^{n+1} C(\alpha,k)} \\ = & \sum_{k=2}^{\infty} 2(1-\beta)(2\beta-1)^{k-2} \geq 1, \end{aligned}$$

for $b \neq 0$, complex, $0 < \beta \leq \frac{1}{2}$, $n \in N_0$, and $0 \leq \alpha < 1$, and also

$$\sum_{k=2}^{\infty} \frac{k^{n+1} C(\alpha, k)}{|b|} |a_k| = \sum_{k=2}^{\infty} \frac{k^{n+1} C(\alpha, k)}{|b|} \frac{2\beta |b| (2\beta - 1)^{k-2}}{k^{n+1} C(\alpha, k)}$$
$$= \sum_{k=2}^{\infty} 2\beta (2\beta - 1)^{k-2} \geq 1,$$

for $b \neq 0$, complex, $\frac{1}{2} \leq \beta \leq 1$, $n \in N_0$, $0 \leq \alpha < 1$ and $z \in U$.

Corollary 2. *Let the function $f(z)$ defined by* (1.1) *be analytic in U. If for $b \neq 0$, complex, $n \in N_0$, and $0 \leq \alpha < 1$,*

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) |a_k| \leq |b|,$$

then $f(z)$ belongs to $R_{\alpha}^{n}(b)$.

Corollary 3. *Let the function $f(z)$ defined by* (1.1) *be analytic in U. If for $b \neq 0$, complex, $n \in N_0$, and $0 \leq \alpha < 1$,*

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) |a_k| \leq (2\delta - 1) |b|,$$

whenever $\frac{1}{2} < \delta \leq 1$, and

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) |a_k| \leq |b|,$$

whenever $\delta \geq 1$, then $f(z)$ belongs to $R_{\alpha}^{n}(b, \delta)$.

Corollary 4. *Let the function $f(z)$ defined by* (1.1) *be analytic in U. If for $n \in N_0$, $0 \leq \alpha < 1$, $|\lambda| < \frac{\pi}{2}$, and $0 \leq \rho < 1$*

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) |a_k| \leq (2\delta - 1)(1 - \rho) \cos \lambda,$$

whenever $\frac{1}{2} < \delta \leq 1$, and

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) |a_k| \leq (1 - \rho) \cos \lambda,$$

whenever $\delta \geq 1$, then $f(z)$ belongs to $R_{\alpha}^{*n,\lambda}(\rho)$.

Corollary 5. *Let the function $f(z)$ defined by* (1.1) *be analytic in U. If for $n \in N_0$, $0 \leq \alpha < 1$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \rho < 1$ and $0 \leq \xi < 1$,*

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) |a_k| \leq \left(\frac{1 - \xi}{\xi} \right) (1 - \rho) \cos \lambda,$$

whenever $\frac{1}{2} \le \xi < 1$, *and*

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) |a_k| \le (1-\rho) \cos\lambda,$$

whenever $0 \le \xi \le \frac{1}{2}$, *then* $f(z)$ *belongs to* $R_{\alpha,\delta}^{*n,\lambda}(\xi, \rho)$.

Corollary 6. *Let the function* $f(z)$ *defined by* (1.1) *be analytic in* U. *If for* $n \in N_0$, $0 \le \alpha < 1$, $|\lambda| < \frac{\pi}{2}$, $0 \le \rho < 1$ *and* $0 < \sigma \le 1$,

$$\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) |a_k| \le (1-\rho) \sigma \cos\lambda,$$

then $f(z)$ *belongs to* $\left(R_{\alpha,1}^{n,\lambda}(\rho)\right)^{\sigma}$.

3. COEFFICIENT ESTIMATES

Theorem 7. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $R_{\alpha}^{n}(b, \beta)$. *Then*

$$|a_k| \le \frac{2\beta |b|}{k^{n+1} C(\alpha, k)} \quad (k \ge 2). \tag{3.1}$$

The result is sharp.

Proof. Since $f(z) \in R_{\alpha}^{n}(b, \beta)$, we have from Schwarz's lemma [17]

$$(D^n f * S_\alpha)'(z) = \frac{1 + (2\beta - 1 - 2\beta b)\,\omega(z)}{1 + (2\beta - 1)\,\omega(z)}, \tag{3.2}$$

where $\omega(z) = \sum_{k=1}^{\infty} t_k z^k \in \Omega$. From (3.2), we have

$$\left[2\beta b + (2\beta - 1) \sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_k z^{k-1}\right] \left[\sum_{k=1}^{\infty} t_k z^k\right]$$
$$= -\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_k z^{k-1}. \tag{3.3}$$

Equality corresponding coefficients on both sides of (3.3) we find that the coefficient a_k on the right of (3.3) depends only on $a_2, a_3, \ldots, a_{k-1}$ on the left of (3.3). Therefore, for $k \ge 2$, (3.3) yields

$$\left[2\beta b + (2\beta - 1) \sum_{k=2}^{m-1} k^{n+1} C(\alpha, k) a_k z^{k-1}\right] \omega(z)$$
$$= -\sum_{k=2}^{m} k^{n+1} C(\alpha, k) a_k z^{k-1} - \sum_{k=m+1}^{\infty} b_k z^{k-1}, \tag{3.4}$$

where $\sum\limits_{k=m+1}^{\infty} b_k z^{k-1}$ converges in U. Then, since $|\omega(z)| < 1$ by using Parseval's identity [17], we obtain

$$\sum_{k=2}^{m} k^{2(n+1)} \left(C(\alpha,k)\right)^2 |a_k|^2 r^{2(k-1)} + \sum_{k=m+1}^{\infty} |b_k|^2 r^{2(k-1)}$$

$$\leq \quad 4\beta^2 |b|^2 + (2\beta-1)^2 \sum_{k=2}^{m-1} k^{2(n+1)} \left(C(\alpha,k)\right)^2 |a_k|^2 r^{2(k-1)}. \tag{3.5}$$

Taking the limit as r approaches 1, we have

$$\sum_{k=2}^{m} k^{2(n+1)} \left(C(\alpha,k)\right)^2 |a_k|^2$$

$$\leq \quad 4\beta^2 |b|^2 + (2\beta-1)^2 \sum_{k=2}^{m-1} k^{2(n+1)} \left(C(\alpha,k)\right)^2 |a_k|^2 .$$

Thus

$$m^{2(n+1)} \left(C(\alpha,m)\right)^2 |a_m|^2$$

$$\leq \quad 4\beta^2 |b|^2 + (2\beta-1)^2 \sum_{k=2}^{m-1} k^{2(n+1)} \left(C(\alpha,k)\right)^2 |a_k|^2 . \tag{3.6}$$

Since $0 < \beta \leq 1$, (3.6) yields $m^{2(n+1)} \left(C(\alpha,m)\right)^2 |a_m|^2 \leq 4\beta^2 |b|^2$ which implies $|a_m| \leq \frac{2\beta|b|}{m^{n+1}C(\alpha,m)} \quad (m \geq 2)$. □

The following example shows that the inequality (3.1) is sharp.

Example 1. *Let*

$$\left(D^n f * S_\alpha\right)(z) = \int_0^z \frac{1-(2\beta-1-2\beta b)\, t^{k-1}}{1-(2\beta-1)\, t^{k-1}} \mathrm{d}t, \tag{3.7}$$

where $b \neq 0$, complex, $0 < \beta \leq 1$, $n \in N_0$ and $0 \leq \alpha < 1$. Then it is easy to see that

$$\left| \frac{\left(D^n f * S_\alpha\right)'(z) - 1}{2\beta\left[\left(D^n f * S_\alpha\right)'(z) - 1 + b\right] - \left[\left(D^n f * S_\alpha\right)'(z) - 1\right]} \right| < 1 \quad (z \in U),$$

which proves that $f(z) \in R_\alpha^n(b,\beta)$. Then the function $\left(D^n f * S_\alpha\right)(z)$ has the expansion

$$\left(D^n f * S_\alpha\right)(z) = z + \frac{2\beta b}{k} z^k + \ldots \quad (z \in U),$$

which shows that the estimate (3.1) is sharp.

On replacing the pair (b,β), in turn, by the pairs $(b,1)$, $\left(b,\frac{2\delta-1}{2\delta}\right)$ $\left(\delta>\frac{1}{2}\right)$, $\left((1-\rho)\,\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda,\frac{2\delta-1}{2\delta}\right)$ $\left(|\lambda|<\frac{\pi}{2},\,0\le\rho<1,\,\delta>\frac{1}{2}\right)$, $\left((1-\rho)\,\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda,1-\xi\right)$ $\left(0\le\rho<1,|\lambda|<\frac{\pi}{2},0\le\xi<1\right)$ and $\left((1-\rho)\,\sigma\mathrm{e}^{-\mathrm{i}\lambda}\cos\lambda,\frac{1}{2}\right)$ $\left(|\lambda|<\frac{\pi}{2},\,0\le\rho<1,0<\sigma\le1\right)$ in Theorem 2 we obtain, respectively, the coefficient estimates for the classes $R_\alpha^n(b)$, $R_\alpha^n(b,\delta)$, $R_{\alpha,\delta}^{*n,\lambda}(\rho)$, $R_\alpha^{*n,\lambda}(\xi,\rho)$ and $\left(R_{\alpha,1}^{n,\lambda}(\rho)\right)^\sigma$; which we state in the following corollaries:

Corollary 8. *Let the function $f(z)$ defined by (1.1) be in $R_\alpha^n(b)$. Then $|a_k|\le\frac{2|b|}{kC(\alpha,k)}$ $(k\ge2)$. The result is sharp.*

Corollary 9. *Let the function $f(z)$ defined by (1.1) be in $R_\alpha^n(b,\delta)$. Then $|a_k|\le\frac{\left(\frac{2\delta-1}{\delta}\right)|b|}{k^{n+1}C(\alpha,k)}$ $(k\ge2)$. The result is sharp.*

Corollary 10. *Let the function $f(z)$ defined by (1.1) be in $R_{\alpha,\delta}^{*n,\lambda}(\rho)$. Then $|a_k|\le\left(\frac{2\delta-1}{\delta}\right)\frac{(1-\rho)\cos\lambda}{k^{n+1}C(\alpha,k)}$ $(k\ge2)$. The result is sharp.*

Corollary 11. *Let the function $f(z)$ defined by (1.1) be in $R_\alpha^{*n,\lambda}(\xi,\rho)$. Then $|a_k|\le\frac{2(1-\xi)}{k^{n+1}C(\alpha,k)}(1-\rho)\cos\lambda$ $(k\ge2)$. The result is sharp.*

Corollary 12. *Let the function $f(z)$ defined by (1.1) be in $\left(R_{\alpha,1}^{*n,\lambda}(\rho)\right)^\sigma$. Then $|a_k|\le\frac{(1-\rho)\sigma\cos\lambda}{k^{n+1}C(\alpha,k)}$ $(k\ge2)$. The result is sharp.*

Remark 1. By taking appropriate values of b, β, n, and α in Theorem 2 we obtain the corresponding results established by Makówka [14], Padmanabhan [19], Caplinger and Causey [3], Goel [7], MacGregor [12,13], Ahuja [1], Chen [4], Owa [18], Mogra [15], Gopalakrishna and Umarani [9], and Juneja and Mogra [10].

4. MAXIMIZATION OF $\left|A_3-\mu A_2^2\right|$

We shall need in our discussion the following lemma [11]:

Lemma 13. *Let $\omega(z)=\sum\limits_{m=1}^{\infty}t_mz^m\in\Omega$, if μ is any complex number, then*

$$\left|t_2-\mu t_1^2\right|\le\max\{1,|\mu|\}. \tag{4.1}$$

Equality may be attained with the functions $\omega(z)=z^2$ and $\omega(z)=z$ for $|\mu|<1$ and $|\mu|\ge1$, respectively.

Theorem 14. *Let the function $f(z)$ defined by (1.1) be in the class $R_\alpha^n(b,\beta)$, $\beta\ne\frac{1}{2}$, then for any complex number μ, we have*

$$\left|a_3-\mu a_2^2\right|\le\frac{2\beta|b|}{3^{n+1}C(\alpha,3)}\max\{1,|d|\}, \tag{4.2}$$

where

$$d=\frac{-2^{2n+1}\left(C\left(\alpha,2\right)\right)^{2}\left(2\beta-1\right)+3^{n+1}C\left(\alpha,3\right)\mu\beta b}{2^{2n+1}\left(C\left(\alpha,2\right)\right)^{2}}. \tag{4.3}$$

The result is sharp.

Proof. Since $f\left(z\right)\in R_{\alpha}^{n}\left(b,\beta\right)$, we have

$$\left(D^{n}f*S_{\alpha}\right)'\left(z\right)=\frac{1+\left(2\beta-1-2\beta b\right)\omega\left(z\right)}{1+\left(2\beta-1\right)\omega\left(z\right)}, \tag{4.4}$$

where $\omega\left(z\right)=\sum\limits_{m=1}^{\infty}t_{m}z^{m}\in\Omega$. From (4.4), we get

$$\begin{aligned}\omega\left(z\right) &= -\frac{\left(D^{n}f*S_{\alpha}\right)'\left(z\right)-1}{2\beta\left[\left(D^{n}f*S_{a}\right)'\left(z\right)-1+b\right]-\left[\left(D^{n}f*S_{\alpha}\right)'\left(z\right)-1\right]}\\ &= -\frac{1}{2\beta b}\Bigg\{2^{n+1}C\left(\alpha,2\right)a_{2}z+\\ &\qquad \left[3^{n+1}C\left(\alpha,3\right)a_{3}+\frac{2^{2(n+1)}\left(C\left(\alpha,2\right)\right)^{2}\left(1-2\beta\right)}{2\beta b}a_{2}^{2}\right]z^{2}+\ldots\Bigg\}. \end{aligned} \tag{4.5}$$

Now compare the coefficients of z and z^2 on both sides of (4.5). We thus obtain

$$a_{2}=\frac{-\beta b}{2^{n}C\left(\alpha,2\right)}t_{1}, \tag{4.6}$$

and

$$a_{3}=\frac{2\beta b}{3^{n+1}C\left(\alpha,3\right)}\left[\left(2\beta-1\right)t_{1}^{2}-t_{2}\right]. \tag{4.7}$$

Using (4.6), (4.7) and (4.1), we get the result. Since (4.1) is sharp, (4.2) is also sharp. □

REMARK 2. Taking appropriate values of b and β in Theorem 3, we get the corresponding results for the classes $R_{\alpha}^{n}\left(b\right)$, $R_{\alpha}^{n}\left(b,\delta\right)$, $R_{\alpha,\delta}^{*n,\lambda}\left(\rho\right)$, $R_{\alpha}^{*n,\lambda}\left(\xi,\rho\right)$ and $\left(R_{\alpha,1}^{n,\lambda}\left(\rho\right)\right)^{\sigma}$.

5. DISTORTION THEOREM

We now turn to an investigation of distortion properties of $R_{\alpha}^{n}\left(b,\beta\right)$.

THEOREM 15. *Let the function* $f\left(z\right)$ *defined by* (1.1) *be in the class* $R_{\alpha}^{n}\left(b,\beta\right)$. *Then for* $\beta\neq\frac{1}{2}$ *and* $z\in U$,

$$\left|\left(D^{n}f*S_{\alpha}\right)\left(z\right)\right| \leq \int_{0}^{|z|}\frac{1+2\beta\left|b\right|t+\left(2\beta-1\right)\left[2\beta\mathrm{Re}\left(b\right)-\left(2\beta-1\right)\right]t^{2}}{1-\left(2\beta-1\right)^{2}t^{2}}\mathrm{d}t, \tag{5.1}$$

and

$$(5.2)\left|\left(D^n f * S_\alpha\right)(z)\right| \geq \int_0^{|z|} \frac{1-2\beta\,|b|\,t+(2\beta-1)\left[2\beta\mathrm{Re}\,(b)-(2\beta-1)\right]t^2}{1-(2\beta-1)^2\,t^2}\mathrm{d}t.$$

For $\beta=\frac{1}{2}$, the above estimates reduce to $\left|\left(D^n f * S_\alpha\right)(z)\right| \leq r+\frac{|b|}{2}r^2$ and $\left|\left(D^n f * S_\alpha\right)(z)\right| \geq r-\frac{|b|}{2}r^2$ $(|z|=r)$. The bounds are sharp.

Proof. Since $f(z)\in R_\alpha^n(b,\beta)$ we observe that the condition (1.14) coupled with an application of Schwarz's lemma [17], implies $\left|\left(D^n f * S_\alpha\right)'(z)-\zeta\right| < \Re$, where

$$(5.3)\ \zeta=\frac{1-(2\beta-1)\left[2\beta-1-2\beta\mathrm{Re}\,(b)\right]r^2+2\mathrm{i}\beta\,(2\beta-1)\,\mathrm{Im}\,(b)\,r^2}{1-(2\beta-1)^2\,r^2},$$

and

$$(5.4)\qquad \Re=\frac{2\beta\,|b|\,r}{1-(2\beta-1)^2\,r^2}\qquad (|z|=r)\,.$$

Hence we have

$$(5.5)\qquad \frac{1-2\beta\,|b|\,r+(2\beta-1)\left[2\beta\mathrm{Re}(b)-(2\beta-1)\right]r^2}{1-(2\beta-1)^2\,r^2}$$

$$\leq \mathrm{Re}\left\{\left(D^n f * S_\alpha\right)'(z)\right\} \leq \frac{1+2\beta\,|b|\,r+(2\beta-1)\left[2\beta\mathrm{Re}(b)-(2\beta-1)\right]r^2}{1-(2\beta-1)^2\,r^2}.$$

If

$$g(z)=\frac{1+2\beta\,|b|\,z+(2\beta-1)\left[2\beta\mathrm{Re}(b)-(2\beta-1)\right]z^2}{1-(2\beta-1)^2\,z^2},\ \beta\neq\frac{1}{2},$$

then, since $g(0)=1=\left(D^n f * S_\alpha\right)'(z)\mid_{z=0}$ and $g(z)$ is univalent in U, it follows that $\left(D^n f * S_\alpha\right)'(z)$ is subordinate to $g(z)$. Hence

$$(5.6)\ \left|\left(D^n f * S_\alpha\right)'(z)\right| \leq \frac{1+2\beta\,|b|\,r+(2\beta-1)\left[2\beta\mathrm{Re}(b)-(2\beta-1)\right]r^2}{1-(2\beta-1)^2\,r^2}.$$

In view of

$$|f(z)|=\left|\int_0^z f'(s)\,\mathrm{d}s\right| \leq \int_0^{|z|}\left|f'\left(t\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}t,$$

and with the aid of (5.6) we may write

$$\left|\left(D^n f * S_\alpha\right)(z)\right| \leq \int_0^{|z|} \frac{1+2\beta\,|b|\,t+(2\beta-1)\left[2\beta\mathrm{Re}(b)-(2\beta-1)\right]t^2}{1-(2\beta-1)^2\,t^2}\mathrm{d}t.$$

Further, by using (5.5) we obtain

$$\begin{aligned} \left|(D^n f * S_\alpha)(z)\right| &\geq \int_0^{|z|} \mathrm{Re}\left((D^n f * S_\alpha)'\left(te^{i\theta}\right)\right) dt \\ &\geq \int_0^{|z|} \frac{1-2\beta|b|t+(2\beta-1)[2\beta\mathrm{Re}(b)-(2\beta-1)]t^2}{1-(2\beta-1)^2t^2} dt. \end{aligned}$$

The following example shows that the inequalities (5.1) and (5.2) are sharp. □

EXAMPLE 2. *Let*

$$(D^n f * S_\alpha)(z) = \int_0^z \frac{1+2\beta|b|t+(2\beta-1)[2\beta\mathrm{Re}(b)-(2\beta-1)]t^2}{1-(2\beta-1)^2t^2} dt, \tag{5.7}$$

where $b \neq 0$, *complex,* $0 < \beta \leq 1$, $\beta \neq \frac{1}{2}$, $n \in N_0$ *and* $0 \leq \alpha < 1$. *It is easy to verify that* $f(z) \in R_\alpha^n(b,\beta)$, *and that the equalities in (5.1) and (5.2) are attained for* $z = \pm r$.

REMARK 3. (1) Taking appropriate values of b and β in Theorem 4, we get the distortion theorems for functions in the classes $R_\alpha^n(b)$, $R_\alpha^{*n}(\rho)$, $R_\alpha^n(b,\delta)$, $R_{\alpha,\delta}^{*n,\lambda}(\rho)$, $R_\alpha^{*n,\lambda}(\xi,\rho)$, $\left(R_{\alpha,1}^{n,\lambda}(\rho)\right)^\sigma$, $R_\alpha^{n,\lambda}(a,d)$, and $R_{\alpha,m,M}^{n,\lambda}(\rho)$.

(2) The result in Theorem 4 can be used to solve the problem concerning the radii of $R_\alpha^n(b,\beta)$ in $R_\alpha^n(1,1) = R_\alpha^n$.

THEOREM 16. *Let* $n \in N_0$. *If* $f(z) \in R_\alpha^n(b,\beta)$, $\beta \neq \frac{1}{2}$, *then* $f(z) \in R_\alpha^n(1,1) = R_\alpha^n$ *for* $|z| < \acute{r}$, *where*

$$\acute{r} = \frac{1}{\beta|b| + \sqrt{\beta^2|b|^2 - (2\beta-1)[1-2\beta+2\beta\mathrm{Re}(b)]}}.$$

This result is sharp. An extremal function is given in (5.7).

Proof. Let $f(z) \in R_\alpha^n(b,\beta)$. Then according to Theorem 4 for $|z| = r < 1$, $(D^n f * S_\alpha)'(z)$ lies in the closed disc with the center at the point $\frac{1-(2\beta-1)[2\beta-1-2\beta b]r^2}{1-(2\beta-1)^2r^2}$ and radius $\frac{2\beta|b|r}{1-(2\beta-1)^2r^2}$. It can be shown that this disc lies in the right-half plane if $r < \acute{r}$. This completes the proof of Theorem 5. □

6. AN ARGUMENT THEOREM

THEOREM 17. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $R_\alpha^n(b,\beta)$. *Then for* $|z| = r$, $0 \leq r < 1$,

$$\left|\arg(D^n f * S_\alpha)'(z)\right| \leq \sin^{-1}\left\{\frac{2\beta|b|r^2}{\sqrt{a^2+d^2}}\right\}, \tag{6.1}$$

where $a = 1 - (2\beta - 1)[2\beta - 1 - 2\beta \mathrm{Re}(b)]\, r^2$ *and* $d = 2\beta(2\beta - 1)\,\mathrm{Im}(b)\, r^2$. *The result is sharp.*

Proof. By using the similar arguments as in the proof of Theorem 4, it follows that $(D^n f * S_\alpha)'(z)$ assumes values in the circle of Appolonius whose center is at the point ζ and radius is $\Re$, where ζ and $\Re$ are given by (5.3) and (5.4), respectively. Thus $\left|\arg (D^n f * S_\alpha)'(z)\right|$ attains its maximum at points where a ray from the origin is tangent to the circle that is, when

$$\arg (D^n f * S_\alpha)'(z) = \pm \sin^{-1}\left\{\frac{2\beta |b|\, r}{\sqrt{a^2 + d^2}}\right\},$$

where a and d are given as above. The equality in (6.1) holds for the function of the form

$$(D^n f * S_\alpha)(z) = \int_0^z \frac{1 - \eta[2\beta - 1 - 2\beta b]\, t}{1 - (2\beta - 1)\,\eta t} \mathrm{d}t,$$

with suitably chosen η, where $|\eta| = 1$. □

REMARK 4. For suitable values of b and β we obtain the argument theorems for functions in the classes $R_\alpha^n(b)$, $R_\alpha^n(b,\delta)$, $R_{\alpha,\delta}^{*n,\lambda}(\rho)$, $R_\alpha^{*n,\lambda}(\xi,\rho)$ and $\left(R_{\alpha,1}^{n,\lambda}(\rho)\right)^\sigma$.

7. CONVEX SET

THEOREM 18. *If* $f(z)$ *and* $g(z)$ *belong to the class* $R_\alpha^n(b)$*, then* $tf(z) + (1-t)\,g(z)$, $0 \le t \le 1$, *belongs to the class* $R_\alpha^n(b)$.

Proof. Since $f(z)$ and $g(z)$ belong to the class $R_\alpha^n(b)$, we have

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left((D^n f * S_\alpha)'(z) - 1\right)\right\} > 0, \tag{7.1}$$

and

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left((D^n g * S_\alpha)'(z) - 1\right)\right\} > 0, \tag{7.2}$$

for $b \neq 0$, complex, $n \in N_0$ and $0 \le \alpha < 1$. Using (7.1) and (7.2), it follows that

$$\begin{aligned} & \mathrm{Re}\left\{1 + \tfrac{1}{b}\left[t(D^n f * S_\alpha)'(z) + (1-t)(D^n g * S_\alpha)'(z) - 1\right]\right\} \\ = \; & t\mathrm{Re}\left\{1 + \tfrac{1}{b}\left((D^n f * S_\alpha)'(z) - 1\right)\right\} \\ + \; & (1-t)\,\mathrm{Re}\left\{1 + \tfrac{1}{b}\left((D^n g * S_\alpha)'(z) - 1\right)\right\} > 0, \end{aligned}$$

for all $z \in U$. This proves that $tf(z) + (1-t)\,g(z) \in R_\alpha^n(b)$. □

THEOREM 19. *If* $f(z)$ *and* $g(z)$ *belong to the class* $R_\alpha^n(b,\delta)$*, then* $tf(z) + (1-t)\,g(z)$, $0 \le t \le 1$, *belongs to the class* $R_\alpha^n(b,\delta)$.

Proof. Since $f(z)$ and $g(z)$ belong to the class $R_{\alpha}^{n}(b,\delta)$, we have

$$\left|\frac{b-1+(D^{n}f*S_{\alpha})'(z)}{b}-\delta\right|<\delta, \tag{7.3}$$

and

$$\left|\frac{b-1+(D^{n}g*S_{\alpha})'(z)}{b}-\delta\right|<\delta, \tag{7.4}$$

for $b\neq 0$, complex, $n\in N_0$, $0\leq\alpha<1$ and $\delta>\frac{1}{2}$. Using (7.3) and (7.4), it follows that

$$\begin{aligned}
&\left|\frac{b-1+\left[t\,(D^{n}f*S_{\alpha})'(z)+(1-t)\,(D^{n}g*S_{\alpha})'(z)\right]}{b}-\delta\right|\\
\leq\ & t\left|\frac{b-1+\left[t\,(D^{n}f*S_{\alpha})'(z)\right]}{b}-\delta\right|\\
+\ & (1-t)\left|\frac{b-1+(D^{n}g*S_{\alpha})'(z)}{b}-\delta\right|\\
<\ & t\delta+(1-t)\,\delta=\delta,
\end{aligned}$$

for all $z\in U$. This proves that $tf(z)+(1-t)\,g(z)\in R_{\alpha}^{n}(b,\delta)$. □

The following result can also be proved on the similar lines:

THEOREM 20. *If $f(z)$ and $g(z)$ belong to the class $R_{\alpha}^{*n,\lambda}(\xi,\rho)$, then $tf(z)+(1-t)\,g(z)$, $0\leq t\leq 1$, belongs to the same class $R_{\alpha}^{*n,\lambda}(\xi,\rho)$.*

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Received December 20, 2004

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