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A UNIVALENCE CRITERION FOR ANALYTIC FUNCTIONS IN THE UNIT DISK

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Abstract. In this paper we obtain a univalence criterion involving the logarithmic derivative of $z^2 f'(z)/f^2(z)$, where $f(z) = z + a_2 z^2 + \dots$ is an analytic function in the unit disk.

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1. INTRODUCTION

Let U_r denote the disk $\{z \in \mathbb{C} : \ |z| < r\}$, $r \in (0, 1]$. We denote by A the class of functions A that are analytic in the unit disk $U_1 = U = \{z \in \mathbb{C} : \ |z| < 1\}$ with $f(0) = 0$, $f'(0) = 1$.

Before proving our results we need a brief summary of N.N. Pascu's method of constructing univalence criteria [4].

DEFINITION 1. A function $F : U_r \times \mathbb{C} \to \mathbb{C}$, $F = F(u, v)$ satisfies Pommerenke's conditions in U_r, $r \in (0, 1]$ if:

i) the function $L(z, t) = F(\mathrm{e}^{-t}, \mathrm{e}^t z)$ is analytic in U_r, for all $t \in [0, \infty)$, locally absolutely continuous in $[0, \infty)$, locally uniform with respect to U_r.

ii) the function $G(\mathrm{e}^{-t}, \mathrm{e}^t z)$, where $G(u, v) = \frac{u}{v} \cdot \frac{\partial F}{\partial u}(u, v) / \frac{\partial F}{\partial v}(u, v)$ is analytic in U_r for all $t \in [0, \infty)$ and has an analytic extension in $\overline{U} = \{z \in \mathbb{C} : \ |z| \le 1\}$ for all $t > 0$ and in U for $t = 0$. The analytic extension of the function G is denoted by $H = H(\mathrm{e}^{-t} z, \mathrm{e}^t z)$ and is called the associate function of $F(\mathrm{e}^{-t} z, \mathrm{e}^t z)$.

iii) $\frac{\partial F}{\partial v}(0, 0) \neq 0$ and $\frac{\partial F}{\partial u}(0, 0) / \frac{\partial F}{\partial v}(0, 0) \notin (-\infty, -1]$.

iv) the family of functions $\left\{ F(\mathrm{e}^{-t} z, \mathrm{e}^t z) / \left[\mathrm{e}^{-t} \frac{\partial F}{\partial u}(0, 0) + \mathrm{e}^t \frac{\partial F}{\partial v}(0, 0) \right] \right\}_{t \ge 0}$ forms a normal family in U_r.

THEOREM 1. [4] *Let $F : U_r \times \mathbb{C} \to \mathbb{C}$, $F = F(u, v)$ be a function which satisfies Pommerenke's conditions in U_r and let $H = H(u, v)$ be the associate function of F. If*

$$|H(z, z)| < 1, \ \textit{for all } z \in U$$

and

$$\left| H\left(z, \frac{1}{\bar{z}}\right) \right| \le 1, \ \textit{for all } z \in U \setminus \{0\}$$

then the function $F(\mathrm{e}^{-t}z, \mathrm{e}^{t}z)$ *has an analytic and univalent extension in* U*, for all* $t \in [0, \infty)$*.*

2. SUFFICIENT CONDITIONS FOR UNIVALENCE

The following theorem is a direct application of N.N. Pascu's method [4].

THEOREM 2. *Let* $f \in A$ *and let* α *be a complex number such that* $\operatorname{Re} \alpha > \frac{1}{2}$*. If*

$$\left| \frac{1-\alpha}{\alpha} \left[1 - (1-|z|^2) \frac{zf'(z)}{f(z)} \right] + (1-|z|^2) z \frac{d}{dz} \left[\log \frac{z^2 f'(z)}{f^2(z)} \right] \right| \leq |z|^2 \tag{1}$$

for all $z \in U$*, then the function* f *is univalent in* U*.*

Proof. We define

$$F(u,v) = [f(u)]^{1-\alpha} \left[f(u) + \frac{(v-u)f'(u)}{1-(v-u)\left(\frac{f'(u)}{f(u)} - \frac{1}{u} \right)} \right]^{\alpha}. \tag{2}$$

We shall prove that the function $F(u,v)$ satisfies the conditions of Theorem 1. Let

(3)

$$L(z,t) = F(\mathrm{e}^{-t}z, \mathrm{e}^{t}z) = f(\mathrm{e}^{-t}z) \left[1 + \frac{(\mathrm{e}^{2t}-1)\frac{\mathrm{e}^{-t}zf'(\mathrm{e}^{-t}z)}{f(\mathrm{e}^{-t}z)}}{1-(\mathrm{e}^{2t}-1)\left(\frac{\mathrm{e}^{-t}zf'(\mathrm{e}^{-t}z)}{f(\mathrm{e}^{-t}z)} - 1 \right)} \right]^{\alpha}.$$

Since $f(z) \neq 0$ for all $z \in U \setminus \{0\}$ the function

$$f_1(z,t) = \frac{\mathrm{e}^{-t}zf'(\mathrm{e}^{-t}z)}{f(\mathrm{e}^{-t}z)} = 1 + \dots$$

is analytic in U. The function

$$f_2(z,t) = \frac{\mathrm{e}^{-t}zf'(\mathrm{e}^{-t}z)}{f(\mathrm{e}^{-t}z)} - 1 = a_2 \mathrm{e}^{-t}z + \dots$$

is also analytic in U. There exists $r \in (0,1]$ such that the function

$$f_3(z,t) = 1 + \frac{(\mathrm{e}^{2t}-1)f_1(z,t)}{1-(\mathrm{e}^{2t}-1)f_3(z,t)} = \mathrm{e}^{2t} + \dots$$

is analytic in U_r and $f_3(z,t) \neq 0$ for all $z \in U_r$ and $t \in [0,\infty)$. Thus, we can choose an analytic branch in U_r for the function

$$f_4(z,t) = [f_3(z,t)]^{\alpha} = \mathrm{e}^{2\alpha t} + \dots.$$

It follows that the function

$$L(z,t) = f(\mathrm{e}^{-t}z) f_4(z,t) = \mathrm{e}^{(2\alpha-1)t}z + \dots,$$

is analytic in U_r.

Further calculation shows that

$$\frac{\partial L(z,t)}{\partial t} = -\mathrm{e}^{-t} z \frac{\partial F}{\partial u}(\mathrm{e}^{-t}z, \mathrm{e}^{t}z) + \mathrm{e}^{t} z \frac{\partial F}{\partial v}(\mathrm{e}^{-t}z, \mathrm{e}^{t}z) = a_1(t)z + \dots .$$

We obtain that $\left|\frac{\partial L(z,t)}{\partial t}\right|$ is bounded on $[0, T]$ for any fixed $T > 0$ and $z \in U_r$. Hence, the function $L(z,t)$ is locally absolutely continuous in $[0,\infty)$, locally uniform with respect to U_r.

We have

$$a_1(t) = \mathrm{e}^{-t}\frac{\partial F}{\partial u}(0,0) + \mathrm{e}^{t}\frac{\partial F}{\partial v}(0,0) = \mathrm{e}^{(2\alpha-1)t}$$

and hence $a_1(t) \neq 0$ and $\lim\limits_{t\to\infty} |a_1(t)| = \lim\limits_{t\to\infty} \mathrm{e}^{t\mathrm{Re}\ (2\alpha-1)} = \infty$.

It is easy to check that there exists $K > 0$ such that $|F(\mathrm{e}^{-t}z, \mathrm{e}^{t}z)/a_1(t)| \leq K$, for all $z \in U_r$ and $t \in [0,\infty)$ and hence $\{F(\mathrm{e}^{-t}z, \mathrm{e}^{t}z)/a_1(t)\}_{t\geq 0}$ is a normal family in U_r.

From (2) we obtain

$$\begin{aligned} G(u,v) &= \frac{u}{v}\cdot\frac{\partial F}{\partial u}/\frac{\partial F}{\partial v} \\ &= \frac{1-\alpha}{\alpha}\left[\frac{v}{u} - (v-u)\frac{f'(u)}{f(u)}\right] + (v-u)\left[2\frac{1}{u} - 2\frac{f'(u)}{f(u)} + \frac{f''(u)}{f'(u)}\right]. \end{aligned}$$

It follows that the function $G(\mathrm{e}^{-t}z, \mathrm{e}^{t}z)$ has an analytic extension

$$\begin{aligned} H(\mathrm{e}^{-t}z, \mathrm{e}^{t}z) = &\frac{1-\alpha}{\alpha}\left[\mathrm{e}^{2t} - (\mathrm{e}^{2t}-1)\frac{\mathrm{e}^{-t}zf'(\mathrm{e}^{-t}z)}{f(\mathrm{e}^{-t}z)}\right] \\ &+ (\mathrm{e}^{2t}-1)\left[2 - 2\frac{\mathrm{e}^{-t}zf'(\mathrm{e}^{-t}z)}{f(\mathrm{e}^{-t}z)} + \frac{\mathrm{e}^{-t}zf''(\mathrm{e}^{-t}z)}{f'(\mathrm{e}^{-t}z)}\right]. \end{aligned}$$

We have

$$|H(z,z)| = \left|\frac{1-\alpha}{\alpha}\right| < 1,$$

for all $z \in U$ and $\alpha \in \mathbb{C}$ with $\mathrm{Re}\ \alpha > \frac{1}{2}$, and

$$\begin{aligned} \left|H\left(z, \frac{1}{\bar{z}}\right)\right| = &|\frac{1-\alpha}{\alpha}\cdot\frac{1}{|z|^2}\left[1 - (1-|z|^2)\frac{zf'(z)}{f(z)}\right] \\ &+ \frac{1}{|z|^2}(1-|z|^2)z\frac{d}{dz}\left[\log\frac{z^2f'(z)}{f^2(z)}\right]| \leq 1, \end{aligned}$$

for all $z \in U \setminus \{0\}$.

The conditions of Theorem 1 being satisfied it follows that the function $F(\mathrm{e}^{-t}z, \mathrm{e}^{t}z)$ has an analytic and univalent extension $F_1(\mathrm{e}^{-t}z, \mathrm{e}^{t}z)$ in U for all $t \in [0,\infty)$. In particular, the function $f(z) = F_1(z,z)$ is univalent in U. □

Remark 1. If in Theorem 2 the condition (1) is replaced by the condition

$$\left|\frac{1-\alpha}{\alpha}\left[1-(1-|z|^2)\frac{zf'(z)}{f(z)}\right]+(1-|z|^2)z\frac{\mathrm{d}}{\mathrm{d}z}\left[\log\frac{z^2f'(z)}{f^2(z)}\right]\right|\leq q|z|^2,\quad z\in U,$$

where $q\in(0,1)$, then, by Becker's generalized q-chain theory [1], the function f is univalent in U and has a quasiconformal extension in $\mathbb{C}$.

The following corollaries are specific applications of Theorem 2.

Corollary 1. [3] *If $f\in A$ and*

$$\left|z\frac{\mathrm{d}}{\mathrm{d}z}\left[\log\frac{z^2f'(z)}{f^2(z)}\right]\right|\leq\frac{|z|^2}{1-|z|^2},\quad z\in U,$$

then f is an univalent function in U.

Proof. It follows from Theorem 2 with $\alpha=1$. □

Corollary 2. *If $f\in A$ and*

$$\left|(1-|z|^2)z\frac{\mathrm{d}}{\mathrm{d}z}\left[\log\frac{z^2f'(z)}{f^2(z)}\right]+(1-|z|^2)\frac{zf'(z)}{f(z)}-1\right|\leq|z|^2,\quad z\in U,$$

then the function f is univalent in U.

Proof. It follows from Theorem 2 with $\alpha\to\infty$. □

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