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ON A PARTICULAR FIRST ORDER NONLINEAR DIFFERENTIAL SUBORDINATION. I

GH. OROS and GEORGIA IRINA OROS

Abstract. We find conditions on the complex-valued functions B, C, D in unit disc U and the positive constants M and N such that

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < 1$$

implies $|p(z)| < M$, where p is analytic in U, with $p(0) = 0$.

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1. INTRODUCTION AND PRELIMINARIES

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U],\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,\ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U],\ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots,\ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

We let Q denote the class of functions q that are holomorphic and injective in $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial U : \ \lim_{z \to \zeta} q(z) = \infty\right\}$$

and furthermore $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$, where $E(q)$ is called exception set.

In order to prove the new results we shall use the following:

Lemma A. [1] (Lemma 2.2.d, p. 24) *Let $q \in Q$, with $q(0) = a$, and let*

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 \mathrm{e}^{\mathrm{i}\theta_0} \in U$, $r_0 < 1$ and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$, *and*

(iii) $\mathrm{Re}\ \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \mathrm{Re}\ \left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right].$

In [1] chapter IV, the authors have analyzed a first-order linear differential subordination

$$B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \tag{1}$$

where B, C, D and h are complex-valued functions in the unit disc U. A more general version of (1) is given by

$$B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega, \tag{2}$$

where $\Omega \subset \mathbb{C}$.

In [2] we extended this problem by considering a first order nonlinear differential subordination

$$B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z) \prec h(z) \tag{3}$$

A more general version of (3) is given by:

$$B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z) \in \Omega \tag{4}$$

where $\Omega \subseteq \mathbb{C}$.

The general problem is to find conditions on the complex-valued functions B, C, D and h such that the differential subordination given by (3) and (4) will have dominants and even best dominants.

In this paper we consider the first order nonlinear differential subordination (4) in which $\Omega = \{w;\ |w| < 1\}$ and $E(z) \equiv 0$. Given the functions B, C, D our problem is to find a constant M such that, for $p \in \mathcal{H}[0, n]$, the differential inequality

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < 1$$

implies $|p(z)| < M$.

2. MAIN RESULTS

The results in [2] can certainly be used in the special case when $E(z) \equiv 0$. However, in this case we can improve those results by the following theorem:

THEOREM. *Let $M > 0$, and let n be a positive integer. Suppose that the functions $B, C, D : U \to \mathbb{C}$ satisfy $B(z) \neq 0$, and*

$$\begin{cases} \text{(i) } \operatorname{Re} \dfrac{D(z)}{B(z)} \geq -n \\ \\ \text{(ii) } \left| \operatorname{Im} \dfrac{D(z)}{B(z)} \right| \geq \dfrac{1 + M^2 |C(z)|}{M|B(z)|}. \end{cases} \tag{5}$$

If $p \in \mathcal{H}[0, n]$ and

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < 1 \tag{6}$$

then

$$|p(z)| < M.$$

Proof. If we let

$$W(z) = B(z)zp'(z) + C(z)p^2(z) + D(z)p(z),$$

then

$$|W(z)| = |B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)|. \tag{7}$$

and from (6) and (7) we have

$$|W(z)| < 1, \quad z \in U. \tag{8}$$

Assume that $|p(z)| \not< M$, which is equivalent to $p(z) \not\prec Mz = q(z)$.

According to Lemma A, with $q(z) = Mz$, there exist $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$, $r_0 < 1$, $\theta_0 \in [0, 2\pi)$, $\zeta \in \partial U$, $|\zeta| = 1$ and $m \geq n$, such that $p(z_0) = M\zeta$ and $z_0 p'(z_0) = mM\zeta$, $p(U_{r_0}) \subset q(U) = \Omega$.

Using these conditions in (8) we obtain for $z = z_0$

$$\begin{aligned} |W(z_0| &= |B(z_0)mM\zeta + C(z_0)M^2\zeta^2 + D(z_0)M\zeta| = \\ &= |M[B(z_0)m + D(z_0)] + C(z_0)M^2\zeta| \geq \\ &\geq M|B(z_0)m + D(z_0)| - M^2|C(z_0)|. \end{aligned} \tag{9}$$

Since $m \geq n$ and $B(z) \neq 0$, from condition (5), (i) we have

$$\left| m + \frac{D(z)}{B(z)} \right| \geq \left| n + \frac{D(z)}{B(z)} \right|,$$

and

$$|mB(z) + D(z)| \geq |nB(z) + D(z)|.$$

For $z = z_0$ we deduce

$$|mB(z_0) + D(z_0)| \geq |nB(z_0) + D(z_0)| \geq \left| \frac{D(z_0)}{B(z_0)} \right| \geq \left| \operatorname{Im} \frac{D(z_0)}{B(z_0)} \right|.$$

Using condition (5), (ii) we have

$$\left| n + \frac{D(z_0)}{B(z_0)} \right| \geq \left| \operatorname{Im} \frac{D(z_0)}{B(z_0)} \right| \geq \frac{1 + |C(z_0)|M^2}{M|B(z_0)|},$$

hence

$$|nB(z_0) + D(z_0)| \geq \frac{1 + |C(z_0)|M^2}{M}. \tag{10}$$

Using (10) in condition (9) we have

$$|W(z_0)| \geq M \frac{1 + |C(z_0)|M^2}{M} - M^2|C(z_0)| \geq 1. \tag{11}$$

Since this contradicts (8) we obtain the desired result

$$|p(z)| < M.$$

□

If $C(z)=0$, we obtain Theorem 4.1.c [1], p. 192.

Instead of prescribing the constant M in Theorem in some cases we can use (5), (ii) to determine an appropriate $M=M(n,B,C,D)$ so that (6) implies $|p(z)|<M$. This can be accomplished by solving (ii) for M and taking the supremum of the resulting function over U. Condition (5), (ii) is equivalent to

$$|C(z)|M^2 - M|B(z)|\left|\operatorname{Im}\frac{D(z)}{B(z)}\right| + 1 \le 0.$$

This inequality holds if

$$|B(z)|^2\left|\operatorname{Im}\frac{D(z)}{B(z)}\right|^2 \ge 4|C(z)|. \tag{12}$$

If (12) holds, we let

$$M = \sup_{|z|<1}\frac{|B(z)|\left|\operatorname{Im}\frac{D(z)}{B(z)}\right| - \sqrt{|B(z)|^2\left|\operatorname{Im}\frac{D(z)}{B(z)}\right|^2 - 4|C(z)|}}{2|C(z)|} =$$

$$= \sup_{|z|<1}\frac{2}{|B(z)|\cdot\left|\operatorname{Im}\frac{D(z)}{B(z)}\right| + \sqrt{|B(z)|^2\left|\operatorname{Im}\frac{D(z)}{B(z)}\right|^2 - 4|C(z)|}}.$$

If this supremum is finite, we have the following form of Theorem:

Corollary. *Let $M>0$, and let n be a positive integer. Suppose that $p\in\mathcal{H}[0,n]$ and the functions $B,C,D:U\to\mathbb{C}$, with $B(z)\neq 0$, satisfy*

$$\begin{cases} \operatorname{Re}\dfrac{D(z)}{B(z)} \ge -n \\ M = \sup\limits_{|z|<1}\dfrac{2}{|B(z)|\cdot\left|\operatorname{Im}\frac{D(z)}{B(z)}\right| + \sqrt{|B(z)|^2\left|\operatorname{Im}\frac{D(z)}{B(z)}\right|^2 - 4|C(z)|}} < \infty \end{cases}$$

then

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < 1$$

implies

$$|p(z)| < M.$$

Example. If we let $n=3$, $B(z)=\sqrt{3}+i$, $C(z)=8\sqrt{2}+4i$, $D(z)=4-4\sqrt{3}i$, then $M=0.1$ and we deduce that if $p\in\mathcal{H}[0,3]$ and

$$|(\sqrt{3}+i)zp'(z) + (8\sqrt{2}+4i)p^2(z) + (4-4\sqrt{3}i)p(z)| < 1,$$

then

$$|p(z)| < 0.1.$$

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Department of Mathematics, University of Oradea
410087 Oradea, Romania

Faculty of Mathematics and Computer Sciences
Babeş-Bolyai University
400084 Cluj-Napoca, Romania