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ON A CERTAIN CLASS OF HARMONIC MEROMORPHIC FUNCTIONS

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Abstract. In the present paper we introduce a class of harmonic meromorphic functions in the exterior of the unit disc with the condition $f(\infty) = \infty$ that are univalent and sense preserving, also the properties of this class such as coefficient inequality, distortion, convex combination, convolution are obtained.

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1. INTRODUCTION

Let u, v be real harmonic functions in the simply connected domain Ω, then the continuous function $f = u + \mathrm{i}v$ defined in Ω is said to be harmonic in Ω. If $f = u + \mathrm{i}v$ be harmonic in Ω then there exist analytic functions G, H such that $u = \mathrm{Re}G$ and $v = \mathrm{Im}H$, therefore $f = u + \mathrm{i}v = h + \overline{g}$, where $h = \frac{G+H}{2}, \overline{g} = \frac{\overline{G}-\overline{H}}{2}$ and we call h and g analytic part and co-analytic part of f respectively. The Jacobian of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. Lewy [8], Clunie and Sheil Small [1] have showed that the mapping $z \to f(z)$ is sense preserving and locally injective in Ω if and only if $J_f(z) > 0$ in Ω. The harmonic function $f = h + \overline{g}$ is said to be univalent in Ω if the mapping $z \to f(z)$ is sense preserving functions and injective in Ω. The class of harmonic univalent and sense preserving is denoted by H, consisting of all harmonic functions $f = h + \overline{g}$, where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n$, $|b_1| < 1$ and $f(0) = 0$, $f_z(0) = 1$. This class with some different subclasses is investigated by many authors, for example J. M. Jahangiri [3], [4], J. M. Jahangiri and H. Silverman [5], T. Rosy, B. A. Stephen, K. G. Subramanian and J. M. Jahangiri [9], J. Clunie and T. Sheil Small [1] and others. Also the authors recently have investigated a certain subclass of univalent starlike harmonic functions in [11]. In this paper the authors investigate a certain subclass of harmonic meromorphic functions, this useful class with some subclasses has been investigated by J. M. Jahangiri [7], J. M. Jahangiri and H. Silverman [6], T. Rosy, B. A. Stephen, K.G. Subramanian and J. M. Jahangiri [10]. W. Hengartner and G. Schober [2] considered harmonic sense preserving univalent mappings defined on $\overline{U} = \{z : |z| > 1\}$ that map ∞ to ∞ and showed that such mappings can be represented by $f(z) = A\log|z| + h(z) + \overline{g(z)}$ where $h(z) = \alpha z + \sum_{n=0}^{\infty} a_n z^{-n}$, $g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n}$ are analytic in $\overline{U}$ and $|\alpha| > |\beta| \geq 0$, $A \in \mathbb{C}$, also $\frac{\overline{f_{\overline{z}}}}{f_z}$ is analytic and $\left|\frac{f_{\overline{z}}}{f_z}\right| < 1$. Jahangiri in [7] considered the class Σ_H of all harmonic sense preserving univalent meromorphic mappings $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}, g(z) = \sum_{n=1}^{\infty} b_n z^{-n}, |z| > 1. \tag{1}$$

In this paper we introduce the class $HR_M(\gamma, \lambda)$ consisting of all functions of the form (1) satisfying

$$\operatorname{Re}\left\{(1-\lambda)(1+\mathrm{e}^{\mathrm{i}\alpha})\frac{zf'(z)}{z'f(z)} + \lambda\frac{zf''(z)}{z''f(z)} - \mathrm{e}^{\mathrm{i}\alpha}\right\} \geq \gamma, \tag{2}$$

where $z = r\mathrm{e}^{\mathrm{i}\theta}$, $1 < r < \infty$, α, θ are real numbers and $0 \leq \gamma < 1$, also

$$z' = \frac{\partial(r\mathrm{e}^{\mathrm{i}\theta})}{\partial\theta} = \mathrm{i}z,\ z'' = -z,\ f'(z) = \frac{\partial f(r\mathrm{e}^{\mathrm{i}\theta})}{\partial\theta} = \mathrm{i}zh'(z) - \overline{\mathrm{i}zg'(z)},$$

$$f''(z) = -zh'(z) - z^2h''(z) - \overline{zg'(z)} - \overline{z^2g''(z)}. \tag{3}$$

Also the class of harmonic function $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n},\ g(z) = -\sum_{n=1}^{\infty} b_n z^{-n},\ a_n \geq 0,\ b_n \geq 0 \tag{4}$$

that satisfies (2) is denoted by $\overline{HR}_M(\gamma, \lambda)$. The particular case $\lambda = 0$ is investigated recently by T. Rosy, B. A. Stephen, R. G. Subramanian and J. M. Jahangiri in [9].

2. COEFFICIENT BOUNDS

THEOREM 1. *Let $f = h + \overline{g}$, where h, g are in* (1.1), *if* $|b_1| < \frac{1-\lambda-\gamma}{\gamma}$, $\lambda > \gamma$ *and also if*

$$\sum_{n=1}^{\infty} \frac{(2n+1+\gamma+\lambda n^2)|a_n| + (2n-1-\gamma+\lambda n^2)|b_n|}{1-(1+|b_1|)\gamma-\lambda} \leq 1,$$

then f is sense preserving univalent meromorphic harmonic function and $f \in HR_M(\gamma, \lambda)$.

Proof. For proving that $f \in HR_M(\gamma, \lambda)$ we must show that f satisfies (2) and by the relationships in (3) we show that

$$\operatorname{Re}\left\{\frac{-\mathrm{e}^{\mathrm{i}\alpha}h(z) + (1+\mathrm{e}^{\mathrm{i}\alpha}-\lambda\mathrm{e}^{\mathrm{i}\alpha})zh'(z) + \lambda z^2h''(z) - \mathrm{e}^{\mathrm{i}\alpha}\overline{g(z)}}{h(z)+\overline{g(z)}}\right\}$$

$$+\operatorname{Re}\left\{\frac{(2\lambda-1-\mathrm{e}^{\mathrm{i}\alpha}+\lambda\mathrm{e}^{\mathrm{i}\alpha})\overline{zg'(z)} + \lambda\overline{z^2g''(z)}}{h(z)+\overline{g(z)}}\right\} \geq \gamma$$

or

$$\operatorname{Re}\left\{\frac{(1-\lambda\mathrm{e}^{\mathrm{i}\alpha})z + \sum_{n=1}^{\infty}[\lambda n^2+\lambda n-n+\mathrm{e}^{\mathrm{i}\alpha}(\lambda n-n-1)]a_nz^{-n}}{z+\sum_{n=1}^{\infty}a_nz^{-n}+\sum_{n=1}^{\infty}b_n\overline{z^{-n}}}\right\}$$

$$+\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}[\lambda n^2-\lambda n+n+\mathrm{e}^{\mathrm{i}\alpha}(n-1-\lambda n)]b_n\overline{z^{-n}}}{z+\sum_{n=1}^{\infty}a_nz^{-n}+\sum_{n=1}^{\infty}b_n\overline{z^{-n}}}\right\}\geq\gamma$$

and equivalently by putting

$$\begin{aligned}M(z) &= (1-\lambda\mathrm{e}^{\mathrm{i}\alpha})z+\sum_{n=1}^{\infty}\{[n(\lambda n+\lambda+1)+\mathrm{e}^{\mathrm{i}\alpha}(\lambda n-n-1)]a_nz^{-n}\\ &+ [n(\lambda n-\lambda+1)+\mathrm{e}^{\mathrm{i}\alpha}(n-1-\lambda n)]b_n\overline{z^{-n}}\},\end{aligned}$$

$$N(z)=z+\sum_{n=1}^{\infty}a_nz^{-n}+\sum_{n=1}^{\infty}b_n\overline{z^{-n}}.$$

We show that

$$|M(z)+(1-\gamma)N(z)|-|(1+\gamma)N(z)-M(z)|\geq 0.$$

Now we can write

$$\begin{aligned}&|M(z)+(1-\gamma)N(z)|-|(1+\gamma)N(z)-M(z)|\\ &=|(2-\gamma-\lambda\mathrm{e}^{\mathrm{i}\alpha})z+\sum_{n=1}^{\infty}[1-\gamma+\lambda n^2+\lambda n-n+\mathrm{e}^{\mathrm{i}\alpha}(\lambda n-n-1)]a_nz^{-n}|\\ &+\sum_{n=1}^{\infty}[1-\gamma+\lambda n^2-\lambda n+n+\mathrm{e}^{\mathrm{i}\alpha}(n-1-\lambda n)b_n\overline{z^{-n}}\\ &-|(\gamma+\lambda\mathrm{e}^{\mathrm{i}\alpha})z+\sum_{n=1}^{\infty}[(1+\gamma-\lambda n^2-\lambda n+n-\mathrm{e}^{\mathrm{i}\alpha}(\lambda n-n-1)]a_nz^{-n}\\ &+\sum_{n=1}^{\infty}[1+\gamma-\lambda n^2+\lambda n-n-\mathrm{e}^{\mathrm{i}\alpha}(n-1-\lambda n)]b_n\overline{z^{-n}}|\\ &\geq(2-\gamma-\lambda)|z|-\sum_{n=1}^{\infty}(2n+\gamma+\lambda n^2)|a_n||z|^{-n}\\ &-\sum_{n=1}^{\infty}(2n-\gamma+\lambda n^2)|b_n||z|^{-n}\\ &-(\gamma+\lambda)|z|-\sum_{n=1}^{\infty}(2n+2+\gamma+\lambda n^2)|a_n||z|^{-n}-(\gamma+\lambda)|z|^{-1}|b_1|\\ &-\sum_{n=2}^{\infty}(2n-2-\gamma+\lambda n^2)|b_n||z|^{-n}\\ &=2(1-\gamma-\lambda)|z|-2\sum_{n=1}^{\infty}(2n+1+\gamma+\lambda n^2)|a_n||z|^{-n}-2\gamma|b_1||z|^{-1}\\ &-2\sum_{n=1}^{\infty}(2n-1-\gamma+\lambda n^2)|b_n||z|^{-n}\end{aligned}$$

$$= 2|z| \left[1 - \gamma - \lambda - \sum_{n=1}^{\infty} (2n + 1 + \gamma + \lambda n^2)|a_n||z|^{-n-1} - \gamma|b_1||z|^{-2} \right.$$
$$\left. - \sum_{n=1}^{\infty} (2n - 1 - \gamma + \lambda n^2)|b_n||z|^{-n} \right]$$
$$> 2|z| \left[1 - \lambda - \gamma(1 + |b_1|) - \sum_{n=1}^{\infty} (2n + 1 + \gamma + \lambda n^2)|a_n| \right.$$
$$\left. - \sum_{n=1}^{\infty} (2n - 1 - \gamma + \lambda n^2)|b_n| \right] \geq 0.$$

Now for sense preserving we have

$$|h'(z)| \geq 1 - |\sum_{n=1}^{\infty} n a_n z^{-n-1}| > 1 - \sum_{n=1}^{\infty} n|a_n|$$
$$> 1 - (1 + |b_1|)\gamma - \lambda - \sum_{n=1}^{\infty} n|a_n|$$
$$> 1 - (1 + |b_1|)\gamma - \lambda - \sum_{n=1}^{\infty} (2n + 1 + \gamma + \lambda n^2)|a_n|$$
$$\geq \sum_{n=1}^{\infty} (2n - 1 - \gamma + \lambda n^2)|b_n|$$
$$= (1 - \gamma + \lambda)|b_1| + \sum_{n=2}^{\infty} (2n - 1 - \gamma + \lambda n^2)|b_n|$$
$$> (1 - \gamma + \lambda)|b_1| + \sum_{n=2}^{\infty} n|b_n|$$
$$= (\lambda - \gamma)|b_1| + \sum_{n=1}^{\infty} n|b_n| \geq \sum_{n=1}^{\infty} n|b_n| > |g'(z)|.$$

The function f is also univalent because for $z_1, z_2 \in \overline{U}$ with $z_1 \neq z_2$ we have $h(z_1) - h(z_2) \neq 0$ since h is univalent, thus we have

$$\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| = \left| 1 + \frac{\overline{g(z_1)} - \overline{g(z_2)}}{h(z_1) - h(z_2)} \right| \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right|$$
$$= 1 - \left| \frac{\sum_{n=1}^{\infty} b_n (z_1^{-n} - z_2^{-n})}{z_1 - z_2 + \sum_{n=1}^{\infty} a_n (z_1^{-n} - z_2^{-n})} \right|$$
$$\geq 1 - \frac{\sum_{n=1}^{\infty} |b_n||z_1^{-n} - z_2^{-n}|}{|z_1 - z_2| - \sum_{n=1}^{\infty} |a_n||z_1^{-n} - z_2^{-n}|}$$

$$= 1 - \frac{\sum_{n=1}^{\infty} |b_n| \left|\frac{z_1^{-n} - z_2^{-n}}{z_1 - z_2}\right|}{1 - \sum_{n=1}^{\infty} |a_n| \left|\frac{z_1^{-n} - z_2^{-n}}{z_1 - z_2}\right|}.$$

Now, by letting $z_1, z_2 \to 1$ we have

$$\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} = 1 - \frac{\sum_{n=1}^{\infty} n|b_n|}{1 - \sum_{n=1}^{\infty} n|a_n|}$$

$$> 1 - \frac{|b_1| + \sum_{n=2}^{\infty}(2n - 1 - \gamma + \lambda n^2)|b_n|}{1 - (1 + |b_1|)\gamma - \lambda - \sum_{n=1}^{\infty}(2n + 1 + \gamma + \lambda n^2)|a_n|}$$

$$> 1 - \frac{(\gamma - \lambda)|b_1| + \sum_{n=1}^{\infty}(2n - 1 - \gamma + \lambda n^2)|b_n|}{\sum_{n=1}^{\infty}(2n - 1 - \gamma + \lambda n^2)|b_n|}$$

$$= \frac{(\lambda - \gamma)|b_1|}{\sum_{n=1}^{\infty}(2n - 1 - \gamma + \lambda n^2)|b_n|} > 0.$$

Therefore f is univalent and the proof is complete. □

THEOREM 2. *Let $f = h + \overline{g}$, where h and g have the form given by (4), then $f \in \overline{HR}_M(\gamma, \lambda)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{(2n + 1 + \gamma + \lambda n^2)a_n + (2n - 1 - \gamma + \lambda n^2)b_n}{1 - (1 + b_1)\gamma - \lambda} \leq 1. \tag{5}$$

Proof. Since $\overline{HR}_M(\gamma, \lambda) \subseteq HR_M(\gamma, \lambda)$ the "if" part of theorem is clear, for "only if" part we show that $f \notin \overline{HR}_M(\gamma, \lambda)$ if the inequality in (5) does not hold. Let $f \in \overline{HR}_M(\gamma, \lambda)$, then

$$\mathrm{Re}\left\{\frac{-(\gamma + \mathrm{e}^{\mathrm{i}\alpha})h(z) + (1 + \mathrm{e}^{\mathrm{i}\alpha} - \lambda \mathrm{e}^{\mathrm{i}\alpha})zh'(z) + \lambda z^2 h''(z)}{h(z) + \overline{g(z)}}\right\}$$

$$-\mathrm{Re}\left\{\frac{(\gamma + \mathrm{e}^{\mathrm{i}\alpha})\overline{g(z)} + (2\lambda - 1 - \mathrm{e}^{\mathrm{i}\alpha} + \lambda \mathrm{e}^{\mathrm{i}\alpha})\overline{zg'(z)} + \lambda\overline{z^2 g''(z)}}{h(z) + \overline{g(z)}}\right\} \geq 0$$

or equivalently

$$\mathrm{Re}\left\{\frac{(1 - \gamma - \lambda \mathrm{e}^{\mathrm{i}\alpha})z - \sum_{n=1}^{\infty}[n + \gamma - \lambda n^2 - \lambda n + \mathrm{e}^{\mathrm{i}\alpha}(n + 1 - \lambda n)]a_n z^{-n}}{z + \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \overline{z^{-n}}}\right\}$$

$$-\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}[n - \gamma + \lambda n^2 - \lambda n + \mathrm{e}^{\mathrm{i}\alpha}(n - 1 - \lambda n)]b_n \overline{z^{-n}}}{z + \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \overline{z^{-n}}}\right\} \geq 0.$$

Now, for $|z| = r > 1$ this inequality reduces to

$$\mathrm{Re}\left\{\frac{(1 - \gamma - \lambda \mathrm{e}^{\mathrm{i}\alpha}) - \sum_{n=1}^{\infty}[n + \gamma - \lambda n^2 - \lambda n + \mathrm{e}^{\mathrm{i}\alpha}(n + 1 - \lambda n)]a_n r^{-n-1}}{1 + \sum_{n=1}^{\infty} a_n r^{-n-1} - \sum_{n=1}^{\infty} b_n r^{-n-1}}\right\}$$

$$-\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}[n - \gamma + \lambda n^2 - \lambda n + \mathrm{e}^{\mathrm{i}\alpha}(n - 1 - \lambda n)]b_n r^{-n-1}}{1 + \sum_{n=1}^{\infty} a_n r^{-n-1} - \sum_{n=1}^{\infty} b_n r^{-n-1}}\right\} \geq 0.$$

However we can write:

$$\frac{1-\gamma+\lambda-\sum_{n=1}^{\infty}[(2n+1+\gamma+\lambda n^2)a_n+(2n-1-\gamma+\lambda n^2)b_n]r^{-n-1}}{1+\sum_{n=1}^{\infty}(a_n-b_n)r^{-n-1}}$$

$$\geq \mathrm{Re}\left\{\frac{(1-\gamma-\lambda e^{i\alpha})-\sum_{n=1}^{\infty}[n+\gamma-\lambda n^2+e^{i\alpha}(n+1-\lambda n)]a_n r^{-n-1}}{1+\sum_{n=1}^{\infty}(a_n-b_n)r^{-n-1}}\right\}$$

$$-\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}[n-\gamma+\lambda n^2-\lambda n+e^{i\alpha}(n-1-\lambda n)]b_n r^{-n-1}}{1+\sum_{n=1}^{\infty}(a_n-b_n)r^{-n-1}}\right\}\geq 0.$$

Consequently, we obtain $\frac{M(r)}{N(r)} \geq 0$, where $N(r) = 1+\sum_{n=1}^{\infty}(a_n-b_n)r^{-n-1}$ and

$$M(r)=1-\gamma+\lambda-\sum_{n=1}^{\infty}[(2n+1+\gamma+\lambda n^2)a_n+(2n-1-\gamma+\lambda n^2)b_n]r^{-n-1},$$

also by letting $r \to 1^+$ we have

$$\frac{1-\gamma+\lambda-\sum_{n=1}^{\infty}[(2n+1+\gamma+\lambda n^2)a_n+(2n-1-\gamma+\lambda n^2)b_n]}{1+\sum_{n=1}^{\infty}(a_n-b_n)}\geq 0. \tag{6}$$

Now, by condition (5) we get

$$\sum_{n=1}^{\infty}[(2n+1+\gamma+\lambda n^2)a_n+(2n-1-\gamma+\lambda n^2)b_n]$$
$$\leq 1-(1+b_1)\gamma-\lambda<1-\gamma-\lambda<1-\gamma+\lambda.$$

So, if this condition does not hold then the numerator in (6) is negative and therefore the quotient in (6) is also negative, this contradicts the required condition that (6) must be hold. □

3. EXTREME POINTS AND DISTORTION BOUNDS

THEOREM 3. *Let for* $j=1,2,3,\ldots$ *the functions* $f_j(z)=z+\sum_{n=1}^{\infty}a_{n,j}z^{-n}-\sum_{n=1}^{\infty}b_{n,j}\overline{z^{-n}}$ *belongs to* $\overline{HR}_M(\gamma,\lambda)$, *then* $F(z)=\sum_{j=1}^{\infty}\mu_j f_j(z)$ *belongs to* $\overline{HR}_M(\gamma,\lambda)$, *where*

$$\sum_{j=1}^{\infty}\mu_j=1,\ 0\leq\mu_j<1, j=1,2,3,\ldots.$$

Proof. We have $F(z) = z+\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} \mu_j a_{n,j})z^{-n}-\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} \mu_j b_{n,j})\overline{z^{-n}}$ and also we can write $\sum_{n=1}^{\infty} \frac{(2n+1+\gamma+\lambda n^2)a_{n,j}+(2n-1-\gamma+\lambda n^2)b_{n,j}}{1-(1+b_1)\gamma-\lambda} \leq 1$, therefore

$$\sum_{n=1}^{\infty} \frac{(2n+1+\gamma+\lambda n^2)(\sum_{j=1}^{\infty} \mu_j a_{n,j}) + (2n-1-\gamma+\lambda n^2)(\sum_{j=1}^{\infty} \mu_j b_{n,j})}{1-(1+b_1)\gamma-\lambda}$$
$$= \sum_{j=1}^{\infty}\left[\sum_{n=1}^{\infty} \frac{(2n+1+\gamma+\lambda n^2)a_{n,j} + (2n-1-\gamma+\lambda n^2)b_{n,j}}{1-(1+b_1)\gamma-\lambda}\right]\mu_j$$
$$\leq \sum_{j=1}^{\infty} \mu_j = 1$$

and this shows that $F \in \overline{HR}_M(\gamma, \lambda)$. □

In the next theorem we obtain the extreme points of $\overline{HR}_M(\gamma, \lambda)$.

THEOREM 4. *$f \in \overline{HR}_M(\gamma, \lambda)$ if and only if it can be expressed of the form*

$$f(z) = \sum_{n=0}^{\infty}(s_n h_n(z) + t_n g_n(z)), \tag{7}$$

where $h_0(z) = z$, $h_n(z) = z + \frac{1-(1+b_1)\gamma-\lambda}{2n+1+\gamma+\lambda n^2}z^{-n}$, $g_0(z) = z$, $g_n(z) = z - \frac{1-(1+b_1)\gamma-\lambda}{2n-1-\gamma+\lambda n^2}\overline{z^{-n}}$, $n = 1, 2, 3, \dots$ and $\sum_{n=0}^{\infty}(s_n + t_n) = 1$.

Proof. If the function f is given by (7), then we have

$$\begin{aligned} f(z) &= (s_0+t_0)z + \sum_{n=1}^{\infty}\left(z + \frac{1-(1+b_1)\gamma-\lambda}{2n+1+\gamma+\lambda n^2}z^{-n}\right)s_n \\ &\quad + \sum_{n=1}^{\infty}\left(z - \frac{1-(1+b_1)\gamma-\lambda}{2n-1-\gamma+\lambda n^2}\overline{z^{-n}}\right)t_n \\ &= z + \sum_{n=1}^{\infty}\frac{1-(1+b_1)\gamma-\lambda}{2n+1+\gamma+\lambda n^2}s_n z^{-n} - \sum_{n=1}^{\infty}\frac{1-(1+b_1)\gamma-\lambda}{2n-1-\gamma+\lambda n^2}t_n\overline{z^{-n}}. \end{aligned}$$

Now, by making use of (5) the following inequality shows $f \in \overline{HR}_M(\gamma, \lambda)$:

$$\sum_{n=1}^{\infty}\frac{1}{1-(1+b_1)\gamma-\lambda}\left[(2n+1+\gamma+\lambda n^2)\frac{1-(1+b_1)\gamma-\lambda}{2n+1+\gamma+\lambda n^2}s_n\right.$$
$$\left.+(2n-1-\gamma+\lambda n^2)\frac{1-(1+b_1)\gamma-\lambda}{2n-1-\gamma+\lambda n^2}t_n\right] = \sum_{n=1}^{\infty}(s_n+t_n) < 1.$$

Conversely, let $f \in \overline{HR}_M(\gamma, \lambda)$, then $\sum_{n=1}^{\infty} \frac{(2n+1+\gamma+\lambda n^2)a_n+(2n-1-\gamma+\lambda n^2)b_n}{1-(1+b_1)\gamma-\lambda} \leq 1$. If we set $s_n = \frac{2n+1+\gamma+\lambda n^2}{1-(1+b_1)\gamma-\lambda}a_n$, $t_n = \frac{2n-1-\gamma+\lambda n^2}{1-(1+b_1)\gamma-\lambda}b_n$, $n = 1, 2, 3, \dots$ and

$\sum_{n=1}^{\infty}(s_n+t_n)=1$, then we have

$$\begin{aligned} f(z) &= z+\sum_{n=1}^{\infty}\frac{1-(1+b_1)\gamma-\lambda}{2n+1+\gamma+\lambda n^2}s_n z^{-n}-\sum_{n=1}^{\infty}\frac{1-(1+b_1)\gamma-\lambda}{2n-1-\gamma+\lambda n^2}t_n\overline{z^{-n}} \\ &= z+\sum_{n=1}^{\infty}(h_n(z)-z)s_n-\sum_{n=1}^{\infty}(z-g_n(z))t_n \\ &= (1-\sum_{n=1}^{\infty}(s_n+t_n))z+\sum_{n=1}^{\infty}s_n h_n(z)+\sum_{n=1}^{\infty}t_n g_n(z) \\ &= (s_0+t_0)z+\sum_{n=1}^{\infty}(s_n h_n(z)+t_n g_n(z))=\sum_{n=0}^{\infty}(s_n h_n(z)+t_n g_n(z)). \end{aligned}$$

□

Theorem 5. *Let* $f\in\overline{HR}_M(\gamma,\lambda)$, *then for* $|z|=r>1$ *we have*

$$r-[1-\gamma-(1+b_1)\lambda]r^{-1}\leq|f(z)|\leq r+[1-\gamma-(1+b_1)\lambda]r^{-1}.$$

Proof. Let $f\in\overline{HR}_M(\gamma,\lambda)$, then for $|z|>1$ we have

$$|f(z)|\leq|z|+\sum_{n=1}^{\infty}a_n|z|^{-n}+\sum_{n=1}^{\infty}b_n|z|^{-n}=r+\sum_{n=1}^{\infty}(a_n+b_n)r^{-n}$$

$$<r-r^{-1}\left(\sum_{n=1}^{\infty}a_n+\sum_{1}^{\infty}b_n\right)$$

$$<r+r^{-1}\sum_{n=1}^{\infty}(2n+1+\gamma+\lambda n^2)a_n+r^{-1}b_1+r^{-1}\sum_{n=1}^{\infty}(2n-1-\gamma+\lambda n^2)b_n$$

$$=r+(\gamma-\lambda)r^{-1}b_1$$

$$+r^{-1}\left[\sum_{n=1}^{\infty}(2n+1+\gamma+\lambda n^2)a_n+\sum_{n=1}^{\infty}(2n-1-\gamma+\lambda n^2)b_n\right]$$

$$\leq r+(\gamma-\lambda)r^{-1}b_1+r^{-1}(1-(1+b_1)\gamma-\lambda)=r+[1+\gamma-(1+b_1)\lambda]r^{-1}.$$

Also we obtain

$$|f(z)|\geq r-\sum_{n=1}^{\infty}a_n r^{-n}-\sum_{n=1}^{\infty}b_n r^{-n}>r-r^{-1}\sum_{n=1}^{\infty}a_n-r^{-1}\sum_{n=1}^{\infty}b_n$$

$$>r-r^{-1}\sum_{n=1}^{\infty}(2n+1+\gamma+\lambda n^2)a_n-r^{-1}b_1-r^{-1}\sum_{n=2}^{\infty}(2n-1-\gamma+\lambda n^2)b_n$$

$$=r+(\lambda-\gamma)r^{-1}b_1-r^{-1}\left[\sum_{n=1}^{\infty}(2n+1+\gamma+\lambda n^2)a_n+\sum_{n=1}^{\infty}(2n-1-\gamma+\lambda n^2)b_n\right]$$

$$\geq r+(\lambda-\gamma)r^{-1}b_1-r^{-1}(1-(1+b_1)\gamma-\lambda)=r-[1-\gamma-(1+b_1)\lambda]r^{-1}.$$

□

4. α-CONVOLUTION ON $\overline{HR}_M(\gamma,\lambda)$

DEFINITION. Let $f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \overline{z^{-n}}, F(z) = z + \sum_{n=1}^{\infty} c_n z^{-n} - \sum_{n=1}^{\infty} d_n \overline{z^{-n}}$ be in $\overline{HR}_M(\gamma,\lambda)$ for real number α. We define the α-convolution of f and F as follow

$$(f \otimes_\alpha F)(z) = z + \sum_{n=1}^{\infty} \frac{a_n c_n z^{-n}}{n^\alpha} + \sum_{n=1}^{\infty} \frac{b_n d_n}{n^\alpha}\overline{z^{-n}}.$$

The 0-convolution of f and F is the familiar Hadamard product

$$(f * F)(z) = (f \otimes_0 F)(z) = z + \sum_{n=1}^{\infty} a_n c_n z^{-n} + \sum_{n=1}^{\infty} b_n d_n \overline{z^{-n}},$$

also the 1-convolution of f and F is named integral convolution and is defined by

$$(f \otimes_1 F)(z) = z + \sum_{n=1}^{\infty} \frac{a_n c_n}{n} z^{-n} + \sum_{n=1}^{\infty} \frac{b_n d_n}{n}\overline{z^{-n}}.$$

THEOREM 6. *Let*

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \overline{z^{-n}}, F(z) = z + \sum_{n=1}^{\infty} c_n z^{-n} - \sum_{n=1}^{\infty} d_n \overline{z^{-n}}$$

be in $\overline{HR}_M(\gamma,\lambda)$. Then the α-convolution of f and F, where $\alpha \geq \frac{1-(1+b_1)\gamma-\lambda}{1-\gamma+\lambda}$, belongs to $\overline{HR}_M(\gamma,\lambda)$ if further more of the above condition for α we have

$$\alpha \geq \max_{n\neq 1}\left\{(\log n)^{-1}\log\frac{1-(1+b_1)\gamma-\lambda}{2n+1+\gamma+\lambda n^2}, (\log n)^{-1}\log\frac{1-(1+b_1)\gamma-\lambda}{2n-1-\gamma+\lambda n^2}\right\}.$$

Proof. Since $f, F \in \overline{HR}_M(\gamma,\lambda)$ we can write

$$\sum_{n=1}^{\infty}\left[\frac{2n+1+\gamma+\lambda n^2}{1-(1+b_1)\gamma-\lambda}a_n + \frac{2n-1-\gamma+\lambda n^2}{1-(1+b_1)\gamma-\lambda}b_n\right] \leq 1, \tag{8}$$

$$\sum_{n=1}^{\infty}\left[\frac{2n+1+\gamma+\lambda n^2}{1-(1+b_1)\gamma-\lambda}c_n + \frac{2n-1-\gamma+\lambda n^2}{1-(1+b_1)\gamma-\lambda}d_n\right] \leq 1.$$

Also we have

$$c_n \leq \frac{1-(1+b_1)\gamma-\lambda}{2n+1+\gamma+\lambda n^2},\ d_n \leq \frac{1-(1+b_1)\gamma-\lambda}{2n-1-\gamma+\lambda n^2}$$

and we must show

$$\sum_{n=1}^{\infty}\left[\frac{2n+1+\gamma+\lambda n^2}{1-(1+b_1)\gamma-\lambda}\frac{a_n c_n}{n^\alpha} + \frac{2n-1-\gamma+\lambda n^2}{1-(1+b_1)\gamma-\lambda}\frac{b_n d_n}{n^\alpha}\right] \leq 1. \tag{9}$$

For this purpose we have

$$\sum_{n=1}^{\infty}\left[\frac{(2n+1+\gamma+\lambda n^2)a_n c_n}{[1-(1+b_1)\gamma-\lambda]n^\alpha} + \frac{(2n-1-\gamma+\lambda n^2)b_n d_n}{[(1-(1+b_1)\gamma-\lambda]n^\alpha}\right] \leq \sum_{n=1}^{\infty}\frac{a_n+b_n}{n^\alpha}.$$

Therefore, in view of (8) the inequality in (9) holds true if the both below inequalities hold true.

$$n^{\alpha} \geq \frac{1-(1+b_1)\gamma-\lambda}{2n+1+\gamma+\lambda n^2},\ n^{\alpha} \geq \frac{1-(1+b_1)\gamma-\lambda}{2n-1-\gamma+\lambda n^2}.$$

Equivalently, (9) holds true if

$$\alpha \geq \max_{n\neq 1}\left\{(\log n)^{-1}\log\frac{1-(1+b_1)\gamma-\lambda}{2n+1+\gamma+\lambda n^2}, (\log n)^{-1}\log\frac{1-(1+b_1)\gamma-\lambda}{2n-1-\gamma+\lambda n^2}\right\}.$$

□

THEOREM 7. *Suppose* $0 \leq \gamma_1 \leq \gamma_2 < 1$, $f \in \overline{HR}_M(\gamma_2,\lambda)$ *and* $g \in \overline{HR}_M(\gamma_1,\lambda)$. *Then* $f * g \in \overline{HR}_M(\gamma_2,\lambda) \subset \overline{HR}_M(\gamma_1,\lambda)$.

Proof. By condition $0 \leq \gamma_1 \leq \gamma_2 < 1$ it is clear that $\overline{HR}_M(\gamma_2,\lambda) \subset \overline{HR}_M(\gamma_1,\lambda)$, also we have

$$\sum_{n=1}^{\infty}\frac{(2n+1+\gamma_2+\lambda n^2)a_n+(2n-1-\gamma_2+\lambda n^2)b_n}{1-(1+b_1)\gamma_2-\lambda} \leq 1,$$

$$\sum_{n=1}^{\infty}\frac{(2n+1+\gamma_1+\lambda n^2)c_n+(2n-1-\gamma_1+\lambda n^2)d_n}{1-(1+b_1)\gamma_1-\lambda} \leq 1$$

and consequently we can write

$$c_n \leq \frac{1-(1+b_1)\gamma_1-\lambda}{2n+1+\gamma_1+\lambda n^2},\ d_n \leq \frac{1-(1+b_1)\gamma_1-\lambda}{2n-1-\gamma_1+\lambda n^2},$$

so we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(2n+1+\gamma_2+\lambda n^2)a_nc_n+(2n-1-\gamma_2+\lambda n^2)b_nd_n}{1-(1+b_1)\gamma_2-\lambda}\\
&\leq \sum_{n=1}^{\infty}\frac{(2n+1+\gamma_2+\lambda n^2)(1-(1+b_1)\gamma_1-\lambda)}{(1-(1+b_1)\gamma_2-\lambda)(2n+1+\gamma_1+\lambda n^2)}a_n\\
&+\sum_{n=1}^{\infty}\frac{(2n-1-\gamma_2+\lambda n^2)(1-(1+b_1)\gamma_1-\lambda)}{(1-(1+b_1)\gamma_2-\lambda)(2n-1-\gamma_1+\lambda n^2)}b_n\\
&\leq \sum_{n=1}^{\infty}\frac{2n+1+\gamma_2+\lambda n^2}{1-(1+b_1)\gamma_2-\lambda}a_n+\sum_{n=1}^{\infty}\frac{2n-1-\gamma_2+\lambda n^2}{1-(1+b_1)\gamma_2-\lambda}b_n \leq 1
\end{aligned}$$

and this shows that $f * g \in \overline{HR}_M(\gamma_2,\lambda)$ and the proof is complete. □

In this part we investigate the effect of some special integral operator on the elements of $\overline{HR}_M(\gamma,\lambda)$.

THEOREM 8. *Let* $f \in \overline{HR}_M(\gamma,\lambda)$, *then* $Tf(z) = \int_1^{\infty}(c-1)t^{-c-1}f(tz)\mathrm{d}t \in \overline{HR}_M(\gamma,\lambda)$, *where* $c > 1$.

Proof. By a simple calculation, we have

$$Tf(z) = z + \sum_{n=1}^{\infty} \frac{c-1}{c+n} a_n z^{-n} - \sum_{n=1}^{\infty} \frac{c-1}{c+n} b_n \overline{z^{-n}}.$$

Also, since $f \in \overline{HR}_M(\gamma, \lambda)$, by Theorem 2 we can write

$$\sum_{n=1}^{\infty} \frac{(2n+1+\gamma+\lambda n^2)a_n + (2n-1-\gamma+\lambda n^2)b_n}{1-(1+b_1)\gamma-\lambda} \leq 1. \tag{10}$$

Also, since $\frac{c-1}{c+n} \leq 1$, by considering (10) we obtain

$$\sum_{n=1}^{\infty} \left[\frac{2n+1+\gamma+\lambda n^2}{1-(1+b_1)\gamma-\lambda} \left(\frac{c-1}{c+n}\right) a_n + \frac{2n-1-\gamma+\lambda n^2}{1-(1+b_1)\gamma-\lambda} \left(\frac{c-1}{c+n}\right) b_n \right] \leq 1.$$

Finally, the last inequality shows that $Tf(z) \in \overline{HR}_M(\gamma, \lambda)$. □

Theorem 9. *Let a, b, c be positive real numbers, where $c > \max\{ab, a+b\}$, $a > 1$, for $|z| < 1$ define*

$$F(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^{-n} - \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} \overline{z^{-n}}.$$

Then, for every $f \in \overline{HR}_M(\gamma, \lambda)$,

$$Qf(z) = (F \otimes_\alpha f)(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n! n^\alpha} a_n z^{-n} - \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n! n^\alpha} b_n \overline{z^{-n}}$$

belongs to $\overline{HR}_M(\gamma, \lambda)$ if

$$\alpha \geq \frac{\log(a)_n + \log(b)_n - \log(c)_n - \log n!}{\log n}, \ n \neq 1, (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Proof. By Theorem 2 we have

$$\sum_{n=1}^{\infty} \frac{(2n+1+\gamma+\lambda n^2)a_n + (2n-1-\gamma+\lambda n^2)b_n}{1-(1+b_1)\gamma-\lambda} \leq 1. \tag{11}$$

We must show

$$\sum_{n=1}^{\infty} \left[\frac{(2n+1+\gamma+\lambda n^2)a_n + (2n-1-\gamma+\lambda n^2)b_n}{1-(1+b_1)\gamma-\lambda} \right] \frac{(a)_n(b)_n}{(c)_n n! n^\alpha} \leq 1.$$

Therefore, in view of (11) the inequality in (10) holds true if $n^\alpha \geq \frac{(a)_n(b)_n}{(c)_n n!}$ and this completes the proof. □

Remark 1. All theorems in this paper can be specialized for Rosy's class in [9] by putting $\lambda = 0$ and considering the inequality $1-(1+|b_1|)\gamma-\lambda \leq 1-\gamma$ in Theorem 1 and also in Theorem 2.

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