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CLASSES OF n-STARLIKE FUNCTIONS

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Abstract. In this paper, we define and we investigate several subclasses of n-starlike functions. In particular cases we reobtain some results of Yong Chan Kim and Il Bneg Jung [2].

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1. INTRODUCTION AND DEFINITIONS

Let U denote the open unit disc $U = \{z; z \in \mathbb{C}, |z| < 1\}$, let $H(U)$ denote the class of functions analytic in the unit disc U and let $H_u(U)$ denote the class of functions analytic in U which are univalent in U.

Consider the classes of functions

$$A = \{f \in H(U) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n\}$$

and

$$S = \{f \in H_u(U) : f(0) = f'(0) - 1 = 0\}.$$

For $f \in S$ we define $D^0 f(z) = f(z)$, $D' f(z) = Df(z) = zf'(z)$ and

$$D^n f(z) = D(D^{n-1} f(z)),\ \ n \in \mathbb{N}^* = \{1, 2, 3 \ldots\}.$$

The differential operator D^n was introduced by Sălăgean [9].

With the help of the differential operator D^n Sălăgean [9] introduced the class

$$S_n^*(\alpha) = \left\{ f \in S : \operatorname{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha;\ z \in U \right\},\ \ 0 \le \alpha < 1;\ \ n \in \mathbb{N}.$$

We note that $S_n^*(0) = S_n^*$, $S_{n+1}^*(\alpha) \subset S_n^*(\alpha)$, $n \in \mathbb{N}$, $\alpha \in [0, 1)$, and this gives

$$S_n^*(\alpha) \subset S_{n-1}^*(\alpha) \subset \ldots \subset S_1^*(\alpha) \subset S_0^*(\alpha),$$

where $S_1^*(\alpha) = K(\alpha) \subseteq K(0) = K$ is the class of convex functions and $S_0^* \subseteq S_0^*(0) = S^*$ is the class of starlike functions.

We recall the concept of subordination. Given $f(z)$ and $g(z) \in H(U)$, $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $h(z) \in H(U)$ with $h(0) = 0$ and $|h(z)| < 1$ such that $f(z) = g(h(z))$, $z \in U$. We denote this subordination by: $f(z) \prec g(z)$. In particular, if $g(z) \in H_u(\mathbf{U})$, then $f(z) \prec g(z) \Longleftrightarrow f(0) = g(0)$ and $f(\mathbf{U}) \subset g(\mathbf{U})$.

For $-1 \le B < A \le 1$ Janowski [1] introduced the class $P[A, B]$ consisting of functions $p \in H_u(\mathbf{U})$ with $p(0) = 1$ and $p(z) \prec \dfrac{1 + Az}{1 + Bz}$.

We denote by $S_n^*[A,B]$ the subclass of S consisting of all fuctions $f(z)$ such that $\dfrac{D^{n+1}f(z)}{D^n f(z)} \in P[A,B]$. We have

$$S_0^*[A,B] = S^*[A,B] = \left\{ f \in \ S : \frac{zf'(z)}{f(z)} \in P[A,B] \right\},$$

$$S_1^*[A,B] = S_1^*[A,B] = \left\{ f \in S : \frac{(zf'(z))'}{f'(z)} \in P[A,B] \right\}.$$

Note that $S^*[-1,1] = S^*$ and $K[-1,1] = K$.

Let $\alpha \in \mathbb{R}$. We denote the class of (n,α)-convex functions by $M_{n,\alpha}$, where

$$M_{n,\alpha} = \left\{ f \in S : \mathrm{Re}[(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha \frac{D^{n+2}f(z)}{D^{n+1}f(z)}] > 0; z \in U \right\}.$$

For $n = 0$ this class was defined by P.T. Mocanu in [3].

For $-1 \le B < A \le 1$ and $z \in \mathbf{U}$ we now define the class

$$M_{n,\alpha}(A,B) = \left\{ f \in S : [(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha \frac{D^{n+2}f(z)}{D^{n+1}f(z)}] \prec \frac{1+Az}{1+Bz} \right\}.$$

Then $M_{n,\alpha}(-1,1) = M_{n,\alpha}$, $M_{n,0}(A,B) = S_n^*[A,B]$ and $M_{1,0}(A,B) = K[A,B]$.

Definition 1. Let $c \in \mathbb{C}$ such that Re $c > 0$, and let

$$N = N(c) = [|c|\,(1 + 2\,\mathrm{Re}\, c^{\frac{1}{2}} + \mathrm{Im}\, c]/\,\mathrm{Re}\, c.$$

If $h \in H(\mathbf{U})$, $h(z) = \dfrac{2Nz}{1-z^2}$ and $b = h^{-1}(c)$, then we define the "open door" function Q_c (cf. [7]) as $Q_c(z) = h\left(\frac{z+b}{1+bz}\right)$, $z \in \ \mathbf{U}$.

2. MAIN RESULTS

Applying the method of integral representations (cf. [3]) for functions in $M_{n,\alpha}(A,B)$, $\alpha > 0$, it is not difficult to deduce:

Lemma 1. *The function* f *lies to* $M_{n,\alpha}(A,B)$, $\alpha > 0$, *if and only if there exists a function* $g(z) \in S^*[A,B]$, *such that*

$$D^n f(z) = \left[\frac{1}{\alpha} \int_0^z \{g(t)\}^{\frac{1}{\alpha}} \ t^{-1} \mathrm{d}t \right]^{\alpha}$$

Proof. Setting $g(z) = D^n f(z) \left[\dfrac{D^{n+1}f(z)}{D^n f(z)} \right]^{\alpha}$, such that (1) is satisfied, we observe that

$$\frac{zg'(z)}{g(z)} = (1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha \frac{D^{n+2}f(z)}{D^{n+1}f(z)}$$

Hence $f \in M_{n,\alpha}(A,B) \Longleftrightarrow g \in S^*[A,B]$. $\square$

Theorem 1. *Let $f \in M_{n,\alpha}(A,B)$, $\alpha > 0$ and let*

$$\frac{1+Az}{1+Bz} \prec \alpha Q_{\frac{q}{\alpha}}(z).$$

Then $f \in S_n^*$.

Proof. Since $f \in M_{n,\alpha}(A,B)$, $\alpha > 0$ by using Lemma 1 we deduce that there exists $g \in S^*[A,B]$ such that

$$D^n f(z) = \left[\frac{1}{\alpha} \int_0^z \{g(t)\}^{\frac{1}{\alpha}} t^{-1} \mathrm{d}t \right]^\alpha .$$

By the hypothessis we also have

$$\frac{1}{\alpha}\left(\frac{zg'(z)}{g(z)}\right) \prec \frac{1}{\alpha}\left(\frac{1+Az}{1+Bz}\right) \prec Q_{\frac{1}{\alpha}}(z).$$

Thus, by a result of Miller and Mocanu ([7], Corollary 3.1) we have

$$D^n f(z) = \left[\frac{1}{\alpha} \int_0^z \{g(t)\}^{\frac{1}{\alpha}}\ t^{-1} \mathrm{d}t \right]^\alpha \in S^* \Rightarrow f \in S_n^*.$$

For $n = 0$ this result was obtained by Y.C. Kim and I.B. Jung (1997) [2].

Lemma 2. (Mocanu – 1986, [4]) *Let $P \in H(\mathbf{U})$ satisfying $P \prec Q_c$. If $p \in H(U)$, $p(0) = 1/c$ and $zp'(z) + P(z)p(z) = 1$, then* Re $p(z) > 0$ *in U.*

Making use of Lemma 2 we now prove the next theorem.

Theorem 2. *Let $f \in M_{n,\alpha}(A,B)$, $\alpha > 0$ and if*

$$\frac{D^{n+1}f(z)}{D^n f(z)} + \frac{D^n f(z)}{D^{n+1}f(z)} - 1 \prec Q_1,$$

then $f \in S_n^*[A,B]$.

Proof. If we set $p(z) = \dfrac{D^{n+1}f(z)}{D^n f(z)}$, then $p(z) + \dfrac{zp'(z)}{p(z)} = \dfrac{D^{n+2}f(z)}{D^{n+1}f(z)}$. Hence

$$(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = p(z) + \alpha\frac{zp'(z)}{p(z)}$$

Since $f \in M_{n,\alpha}(A,B)$, we have $p(z) + \alpha\dfrac{zp'(z)}{p(z)} \prec \dfrac{1+Az}{1+Bz}$.

Setting $P(z) = p(z) + \dfrac{1}{p(z)} - 1$, we obtain $zp'(z) + P(z)p(z) = 1$ and $P \prec Q$, by the hypothesis (2). Thus, by Lemma 2, we have $\mathrm{Re}\, p(z) > 0$ ($z \in \mathbf{U}$). Since $\alpha > 0$ we have

$$\mathrm{Re}\left\{\frac{1}{\alpha}\ p(z)\right\} > 0,\ \ (z \in \mathbf{U}). \tag{4}$$

The function $\dfrac{1+Az}{1+Bz}$, with $-1 \le B < A \le 1$, is a convex univalent function in $\mathbf{U}$. Hence, by appealing to a known result of Miller and Mocanu (1981) [5], we conclude from (3) and (4) that :

$$p(z) \prec \frac{1+Az}{1+Bz} \Longleftrightarrow \frac{D^{n+1}f(z)}{D^n f(z)} \prec \frac{1+Az}{1+Bz} \Longrightarrow f \in S_n^*[A,B]$$

and the proof of Theorem 2 is complete. □

As an example of Miller and Mocanu ([6], Corollary 3.2) we consider the case when $\alpha > 0$, $-1 \le B < A \le 1$. The differential equation

$$q(z) + \alpha \frac{zq'(z)}{q(z)} = \frac{1+Az}{1+Bz}$$

has a univalent solution given by

$$q(z) = \begin{cases} \dfrac{z^{\frac{1}{\alpha}}(1+Bz)^{\frac{1}{\alpha}\frac{A-B}{B}}}{\frac{1}{\alpha}\int\limits_0^z t^{\frac{1}{\alpha}-1}\ (1+Bt)^{\frac{1}{\alpha}\frac{A-B}{B}} \mathrm{d}t} & \text{if } B \ne 0 \\ \dfrac{z^{\frac{1}{\alpha}}\ e^{\frac{A}{\alpha}z}}{\frac{1}{\alpha}\int\limits_0^z t^{\frac{1}{\alpha}-1}\ \mathrm{e}^{\frac{A}{\alpha}} \mathrm{d}t} & \text{if } B = 0 \end{cases} \tag{5}$$

If $p(z) \in H(\mathbf{U})$ and satisfies

$$p(z) + \alpha \frac{zp'(z)}{p(z)} \prec \frac{1+Az}{1+Bz},$$

then

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz}. \tag{6}$$

Hence by the equation (3) and (6), we obtain

Theorem 3. *Let $\alpha > 0$ and $f \in M_{n,\alpha}(A,B)$. Then*

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where $q(z)$ is given by (5).

Theorem 4. $S^*_{n+1}(\alpha) \subset M_{n,\alpha}(1-2\alpha,-1)$ $(0 \le \alpha < 1)$.

Proof. If we define $h_\alpha(z) = \dfrac{1+(1-2\alpha)z}{1-z}$ $(\alpha < 1)$, then we can easily see that

$$f \in S^*_{n+1}(\alpha) \Longleftrightarrow \frac{D^{n+2}f(z)}{D^{n+1}f(z)} \prec h_\alpha(z)$$

([9] equation (9)). Hence, by Theorem 1 of [9] we have

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec h_\alpha(z).$$

Therefore we conclude from ([8], Lemma 2.2) that

$$(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)} \prec h_\alpha(z) \;\Rightarrow\; f \in M_{n,\alpha}(1-2\alpha,-1)$$

For $n = 0$ we obtain the result of Y.C. Kim and I.B. Jung [2].

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