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SYMMETRIC LIFTINGS OF QUANTUM LINEAR SPACES

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Abstract. We determine all liftings of quantum linear spaces over the group Hopf algebra of an abelian group that are symmetric as coalgebras. In particular we discuss classes of pointed Hopf algebras that appear in several classification theorems.
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1. INTRODUCTION AND PRELIMINARIES

A large class of pointed Hopf algebras with non-zero integrals was defined in [2] by starting with the group Hopf algebra of an abelian group, then taking Ore extensions by adjoining skew-primitive elements, and finally by factoring a certain Hopf ideal. The same class is obtained with a different approach in [1], where these Hopf algebras are constructed as the liftings of quantum linear spaces over the group algebra of an abelian group. Let us recall the Hopf algebras obtained by these constructions.

We work over a fixed field k. Let C be an abelian group, t, $n_1, \dots, n_t$ positive integers, $c_1, \dots, c_t \in C$, and $\chi_1, \dots \chi_t \in C^*$, where C^* is the character group of C. We also consider $a_1, \dots, a_t \in k$, and $b_{ij} \in k$ for any $1 \leq i < j \leq t$. Assume that the following conditions are satisfied.

- $q_i = \chi_i(c_i)$ is a primitive n_i-th root of unity for any i.
- $\chi_i(c_j) = \chi_j(c_i)^{-1}$ for any $i \neq j$.
- If $a_i = 1$, then $\chi_i^{n_i} = 1$.
- If $c_i^{n_i} = 1$, then $a_i = 0$.
- $b_{ij} = -\chi_i(c_j) b_{ji}$ for any i, j.
- If $b_{ij} \neq 0$, then $\chi_i \chi_j = 1$.
- If $c_i c_j = 1$, then $b_{ij} = 0$.

If we denote by $\mathbf{n} = (n_1, \dots, n_t)$, $\mathbf{c} = (c_1, \dots, c_t)$, $\chi = (\chi_1, \dots, \chi_t)$, $\mathbf{a} = (a_1, \dots, a_t)$, and $\mathbf{b} = (b_{ij})_{1 \leq i < j \leq t}$, then define $H = H(C, \mathbf{n}, \mathbf{c}, \chi, \mathbf{a}, \mathbf{b})$ to be the Hopf algebra generated by the commuting grouplike elements $g \in C$, and the $(1, c_j)$-primitives $x_j, 1 \leq j \leq t$, subject to the relations

$$x_j g = \chi_j(g) g x_j, \; x_j^{n_j} = a_j (c_j^{n_j} - 1),$$
$$x_i x_j = \chi_i(c_j) x_j x_i + b_{ji}(c_j c_i - 1),$$

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for $1 \leq j < i \leq t$. The coalgebra structure is given by

$$\Delta(g) = g \otimes g,\ \varepsilon(g) = 1, \text{ for } g \in C,$$

$$\Delta(x_j) = c_j \otimes x_j + x_i \otimes 1, \text{ for } 1 \leq j \leq t.$$

If $\mathbf{p} = (p_1, \ldots, p_t)$ is a t-tuple of integers, we denote by $\mathbf{c}^{\mathbf{p}} = c_1^{p_1} \ldots c_t^{p_t}$. If all p_i's are non-negative, we also denote $\mathbf{x}^{\mathbf{p}} = x_1^{p_1} \ldots x_t^{p_t}$, with the convention that $x_i^0 = 1$. We also consider the t-tuples $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$. If $\mathbf{d} = (d_1, \ldots, d_t)$ and $\mathbf{p} = (p_1, \ldots, p_t)$ are t-tuples of integers, we say that $\mathbf{d} \leq \mathbf{p}$ if and only if $d_i \leq p_i$ for any i, and in this case we define $\mathbf{p} - \mathbf{d} = (p_1 - d_1, \ldots, p_t - d_t)$.

The set

$$\{g\mathbf{x}^{\mathbf{p}} | g \in C,\ \mathbf{0} \leq \mathbf{p} \leq \mathbf{n} - \mathbf{1}\}$$

is a basis for H.

The quantum binomial formula (see for example [7, Section 5.6]) shows that for any non-negative integer n we have

$$\Delta(x_j^n) = \sum_{0 \leq d \leq n} \binom{n}{d}_{q_j} c_j^d x_j^{n-d} \otimes x_j^d, \tag{1}$$

where $\binom{n}{d}_{q_j}$ is a quantum binomial coefficient. This implies that the comultiplication on a general basis element is given by

$$\Delta(g\mathbf{x}^{\mathbf{p}}) = \sum_{\mathbf{0} \leq \mathbf{d} \leq \mathbf{p}} \alpha_{\mathbf{p},\mathbf{d}} g\mathbf{c}^{\mathbf{d}} \mathbf{x}^{\mathbf{p}-\mathbf{d}} \otimes g\mathbf{x}^{\mathbf{d}}, \tag{2}$$

where the scalars $\alpha_{\mathbf{p},\mathbf{d}}$ are nonzero products of q_j-binomial coefficients and powers of $\chi_j(c_i)$. Note that $\alpha_{\mathbf{p},\mathbf{p}} = \alpha_{\mathbf{p},\mathbf{0}} = 1$.

The aim of this paper is to study which of the Hopf algebras

$$H(C, \mathbf{n}, \mathbf{c}, \chi, \mathbf{a}, \mathbf{b})$$

are symmetric as coalgebras. Symmetric coalgebras were introduced and studied in [6] as a special class of co-Frobenius coalgebras. In the finite dimensional case, symmetric coalgebras are just duals of symmetric algebras. In the infinite dimensional case some completely new aspects show up. A coalgebra C is called symmetric if there exists an injective morphism $\alpha : C \to C^*$ of (C^*, C^*)-bimodules. Equivalent characterizations, involving trace-like linear functionals and non-degenerate symmetric associative bilinear forms are given in [6, Theorem 3.3]. In [6, Theorem 7.1] it is proved that a Hopf algebra H is symmetric as a coalgebra if and only if it is unimodular, i.e. the spaces of left and right integrals on H coincide, and there exists an invertible element u in the dual algebra H^* such that $S^2(h) = \sum u(h_1)u^{-1}(h_3)h_2$ for any $h \in H$, where S is the antipode of H. This last condition is equivalent to the fact that the dual linear map $(S^2)^*$ is an inner automorphism of the algebra H^*. The main aim of this paper is to prove the following.

THEOREM 1. *Let* $H = H(C, \mathbf{n}, \mathbf{c}, \chi, \mathbf{a}, \mathbf{b})$. *The following assertions are equivalent.*

(1) H *is symmetric as a coalgebra.*

(2) $\mathbf{c}^{\mathbf{n-1}} = 1$ *and there exists a map* $u : C \to k^*$ *such that* $u(gc_i) = q_i u(g)$ *for any* $1 \le i \le t$ *and* $g \in C$.

(3) $\mathbf{c}^{\mathbf{n-1}} = 1$ *and for any* $\mathbf{d} \in \mathbf{Z}^t$ *such that* $\mathbf{c}^{\mathbf{d}} = 1$, *we must have* $\mathbf{q}^{\mathbf{d}} = 1$.

We prove the theorem in Section 1. In Section 2 we consider several remarkable examples of such Hopf algebras and perform explicitly the computations indicated by the theorem. For notations and definitions concerning Hopf algebras we refer to [7].

2. PROOF OF THE THEOREM

The left and right integrals on H were computed explicitly in [2].

For $g \in C$, and $\mathbf{0} \le \mathbf{p} \le \mathbf{n-1}$, let $E_{g,\mathbf{p}} \in H^*$ be the map taking $g\mathbf{x}^{\mathbf{p}}$ to 1 and all other basis elements to 0. Then a left integral on H is $E_{(\mathbf{c}^{\mathbf{n-1}})^{-1},\mathbf{n-1}}$, and a right integral on H is $E_{1,\mathbf{n-1}}$, where there is no danger of confusion if we also denote the identity element in C by 1. Therefore H is unimodular if and only if $\mathbf{c}^{\mathbf{n-1}} = 1$.

If we apply two times Equation (2) we find that

$$(I \otimes \Delta)\Delta(g\mathbf{x}^{\mathbf{p}}) = \sum_{\mathbf{0} \le \mathbf{d} \le \mathbf{p}} \sum_{\mathbf{0} \le \mathbf{e} \le \mathbf{d}} \alpha_{\mathbf{p},\mathbf{d}} \alpha_{\mathbf{d},\mathbf{e}} g\mathbf{c}^{\mathbf{d}}\mathbf{x}^{\mathbf{p}-\mathbf{d}} \otimes g\mathbf{c}^{\mathbf{e}}\mathbf{x}^{\mathbf{d}-\mathbf{e}} \otimes g\mathbf{x}^{\mathbf{e}}.$$

We have that $S(x_i) = -c_i^{-1}x_i$, so then $S^2(x_i) = q_i x_i$. Since S^2 is an algebra morphism, this shows that $S^2(g\mathbf{x}^{\mathbf{p}}) = \mathbf{q}^{\mathbf{p}}\mathbf{x}^{\mathbf{p}}$. Therefore, if for some invertible element $u \in H^*$ we have $S^2(h) = \sum u(h_1)u^{-1}(h_3)h_2$ for any $h \in H$, then

$$\sum_{\mathbf{0} \le \mathbf{d} \le \mathbf{p}} \sum_{\mathbf{0} \le \mathbf{e} \le \mathbf{d}} \alpha_{\mathbf{p},\mathbf{d}} \alpha_{\mathbf{d},\mathbf{e}} u(g\mathbf{c}^{\mathbf{d}}\mathbf{x}^{\mathbf{p}-\mathbf{d}}) u^{-1}(g\mathbf{x}^{\mathbf{e}}) g\mathbf{c}^{\mathbf{e}}\mathbf{x}^{\mathbf{d}-\mathbf{e}} = \mathbf{q}^{\mathbf{p}} g\mathbf{x}^{\mathbf{p}}$$

for any g and $\mathbf{p}$. Since $g\mathbf{c}^{\mathbf{e}}\mathbf{x}^{\mathbf{d}-\mathbf{e}} = g\mathbf{x}^{\mathbf{p}}$ if and only if $\mathbf{d} = \mathbf{p}$ and $\mathbf{e} = \mathbf{0}$, we see that

$$u(g\mathbf{c}^{\mathbf{p}})u^{-1}(g) = \mathbf{q}^{\mathbf{p}},$$
$$u(g\mathbf{c}^{\mathbf{d}}\mathbf{x}^{\mathbf{p}-\mathbf{d}})u^{-1}(g\mathbf{x}^{\mathbf{e}}) = 0 \quad \text{for } (\mathbf{d}, \mathbf{e}) \ne (\mathbf{p}, \mathbf{0}).$$

Note that in the first equation we used the fact that $\alpha_{\mathbf{p},\mathbf{p}} = \alpha_{\mathbf{p},\mathbf{0}} = 1$, and in the second equation we used $\alpha_{\mathbf{p},\mathbf{d}} \ne 0$ for any $\mathbf{p}$, $\mathbf{d}$.

Clearly $u(g) \ne 0$ for any g, and $u^{-1}(g) = u(g)^{-1}$. This implies that $u(g\mathbf{c}^{\mathbf{p}}) = \mathbf{q}^{\mathbf{p}}u(g)$. For $\mathbf{e} = \mathbf{0}$ and $\mathbf{d} < \mathbf{p}$ the second equation writes $u(g\mathbf{c}^{\mathbf{d}}\mathbf{x}^{\mathbf{p}-\mathbf{d}}) = 0$, showing that $u(g\mathbf{x}^{\mathbf{d}}) = 0$ for any $\mathbf{0} < \mathbf{d}$. Thus we have showed that an element $u \in H^*$ is invertible and satisfies $S^2(h) = \sum u(h_1)u^{-1}(h_3)h_2$ for any $h \in H$ if and only if $u(g) \ne 0$ for any $g \in C$, $u(g\mathbf{x}^{\mathbf{d}}) = 0$ for any $g \in C$, $\mathbf{0} < \mathbf{d} \le \mathbf{n-1}$, and $u(g\mathbf{c}^{\mathbf{p}}) = \mathbf{q}^{\mathbf{p}}u(g)$ for any $g \in C$, $\mathbf{0} \le \mathbf{p}$. This last condition is equivalent to $u(gc_i) = q_i u(g)$ for any $g \in C$ and $1 \le i \le t$, showing that (1) $\Leftrightarrow$ (2) (note that for (2) we consider the restriction of u to C).

To see that (2) $\Leftrightarrow$ (3) we note that for defining a map $u : C \to k^*$ such that $u(gc_i) = q_i$ for any $g \in C$ and $1 \le i \le t$, or equivalently $u(g\mathbf{c}^{\mathbf{p}}) = \mathbf{q}^{\mathbf{p}} u(g)$ for any $g \in C$, $\mathbf{p} \in \mathbf{Z}^t$, it is enough to define it on each N-coset, where N is the subgroup of C generated by $c_1, \ldots, c_t$. Moreover, to define such a u on a particular N-coset Ng, we set $u(g\mathbf{c}^{\mathbf{p}}) = \mathbf{q}^{\mathbf{p}} u(g)$ for any $\mathbf{p} \in \mathbf{Z}^t$. To make sure that the definition is correct we must check that whenever $\mathbf{c}^{\mathbf{p}} = \mathbf{c}^{\mathbf{e}}$, we must also have $\mathbf{q}^{\mathbf{p}} = \mathbf{q}^{\mathbf{e}}$, and this is clearly equivalent to the condition from (3). □

Corollary 2. *If the Hopf algebra $H = H(C, \mathbf{n}, \mathbf{c}, \chi, \mathbf{a}, \mathbf{b})$ is symmetric as a coalgebra, then $q_1 q_2 \ldots q_t = 1$.*

Proof. We have that $\mathbf{q}^{\mathbf{n-1}} = 1$ since H is unimodular, and $\mathbf{q}^{\mathbf{n}} = 1$ since the order of q_i is n_i for any $1 \le i \le t$. It follows that $q_1 q_2 \ldots q_t = \mathbf{q}^{\mathbf{1}} = 1$. □

3. EXAMPLES

In this section we assume that the basic field k is algebraically closed of characteristic zero. We investigate several examples of Hopf algebras of the form $H = H(C, \mathbf{n}, \mathbf{c}, \chi, \mathbf{a}, \mathbf{b})$. As we mentioned in the introduction, such a Hopf algebra H is pointed, with coradical the group Hopf algebra kC. If we consider the associated graded Hopf algebra gr H, with respect to the coradical filtration, then there exists a split Hopf algebra projection of gr H to kC, and the space R of coinvariants of gr H with respect to the induced coaction of kC on gr H is called the diagram of H. This diagram R is a very interesting object, since it is a braided Hopf algebra, i.e. a Hopf algebra in the braided category of Yetter-Drinfeld modules over kC, and gr H can be reconstructed from R by a bosonization process, i.e. by a certain biproduct of R and kC. Moreover, properties of H can be derived from those of gr H, in particular presentation of gr H by generators and relations can be lifted to H. In fact for our Hopf algebras $H(C, \mathbf{n}, \mathbf{c}, \chi, \mathbf{a}, \mathbf{b})$, the diagram is a quantum linear space generated by $x_1, \ldots, x_t$. If $t = 1$, R is called a quantum line, if $t = 2$, it is called a quantum plane.

3.1. Liftings of quantum lines. If $H = H(C, \mathbf{n}, \mathbf{c}, \chi, \mathbf{a}, \mathbf{b})$ is a lifting of a quantum line, i.e. if $t = 1$, then H can not be symmetric as a coalgebra. Indeed, this follows immediately from Corollary 2, since $\mathbf{q}^{\mathbf{1}} = q_1 \ne 1$ in this case. In fact it is known that these liftings can not be unimodular, so they are not symmetric either. In particular Taft Hopf algebras are not symmetric as coalgebras. More general, any pointed Hopf algebra with abelian coradical of prime index can not be symmetric as a coalgebra (see [1] for the proof of the fact that any such Hopf algebra is a lifting of a quantum line).

3.2. Hopf algebras of dimension 2^m with small coradical. The classification of Hopf algebras of dimension 2^m with coradical kC_2 was done in [8] and with a different method in [3]. It is known that there exists a unique isomorphism type $E(m-1)$ of such a Hopf algebra. For $m \geq 2$, $E(m-1)$ is presented by generators $c, x_1, \ldots, x_{m-1}$ subject to relations

$$c^2 = 1,\ x_i^2 = 0,\ x_i c = -c x_i,\ x_j x_i = -x_i x_j,$$

$$\Delta(c) = c \otimes c,\ \Delta(x_i) = c \otimes x_i + x_i \otimes 1,\ \varepsilon(c) = 1,\ \varepsilon(x_i) = 0,$$

for any $i, j,\ i \neq j$. Thus $E(m-1) = H(C_2, \mathbf{2}, \mathbf{c} = (c, \ldots, c), (\chi, \ldots, \chi), \mathbf{0}, \mathbf{0})$, where $\mathbf{2} = (2, \ldots, 2)$, the character χ is defined by $\chi(c) = -1$, and both $\mathbf{a}$ and $\mathbf{b}$ consist only of zeros. We have that $\mathbf{q} = (-1, \ldots, -1)$. First of all, $E(m-1)$ is unimodular if and only if $\mathbf{c}^{\mathbf{1}} = 1$, i.e. $c^{m-1} = 1$, which happens if and only if m is odd. In this case, if $\mathbf{c}^{\mathbf{d}} = 1$, then $d_1 + \ldots + d_{m-1}$ is even, so then $\mathbf{q}^{\mathbf{d}} = 1$. Therefore $E(m-1)$ is symmetric as a coalgebra if and only if m is odd.

3.3. Pointed Hopf algebras of dimension p^3. The classification of pointed Hopf algebras of dimension p^3, where p is a prime number, was done independently in [1], [3], [9]. We discuss which of these Hopf algebras are symmetric as coalgebras. If such a Hopf algebra is a group Hopf algebra, then it is cosemisimple, so then it is symmetric as a coalgebra. The rest of the class of pointed Hopf algebras of dimension p^3 consists of liftings of quantum lines (the case where the coradical has index p), and liftings of some quantum planes. Therefore the only candidates for being symmetric are these liftings of quantum planes. These are divided in two classes that are described as follows. If λ is a primitive p-th root of 1 and $1 \leq i \leq p-1$ an integer, we denote by $H(\lambda, i)$ the Hopf algebra with generators c, x, y defined by

$$c^p = 1,\ x^p = y^p = 0,\ xc = \lambda cx,\ yc = \lambda^{-i} cy,\ yx = \lambda^{-i} xy,$$

$$\Delta(c) = c \otimes c,\ \Delta(x) = c \otimes x + x \otimes 1,\ \Delta(y) = c^i \otimes y + y \otimes 1.$$

For $p > 2$, we also denote by $H_\delta(\lambda)$ the Hopf algebra with generators c, x, y defined by

$$c^p = 1,\ x^p = y^p = 0,\ xc = \lambda cx,\ yc = \lambda^{-1} cy,\ yx = \lambda^{-1} xy + c^2 - 1,$$

$$\Delta(c) = c \otimes c,\ \Delta(x) = c \otimes x + x \otimes 1,\ \Delta(y) = c \otimes y + y \otimes 1.$$

It is easy to see that all these Hopf algebras are of the form $H(C, \mathbf{n}, \mathbf{c}, \chi, \mathbf{a}, \mathbf{b})$. It is known that any Hopf algebra of dimension p^3 with coradical kC_p is isomorphic either to some $H(\lambda, i)$ or to some $H_\delta(\lambda)$.

Proposition 3. *The Hopf algebra $H(\lambda, i)$ is symmetric as a coalgebra if and only if $i = p-1$. The Hopf algebra $H_\delta(\lambda)$ can not be symmetric as a coalgebra.*

Proof. Let $H = H(\lambda, i)$. We have $\mathbf{c} = (c, c^i)$, $\mathbf{q} = (\lambda, \lambda^{-i^2})$, and $\mathbf{n} = (p, p)$. Thus H is unimodular if and only if $c^{p-1}(c^i)^{p-1} = c^{(p-1)(i+1)} = 1$, and this happens if and only if $i = p-1$. For $i = p-1$, if $\mathbf{d} = (d_1, d_2)$ and $\mathbf{c}^{\mathbf{d}} = 1$, then $c^{d_1 - d_2} = 1$, so p divides $d_1 - d_2$, and then $\mathbf{q}^{\mathbf{d}} = \lambda^{d_1 - d_2} = 1$, and by the Theorem we see that H is symmetric.

For $p \neq 2$ and $H = H_\delta(\lambda)$, we have $\mathbf{c} = (c, c)$, $\mathbf{q} = (\lambda, \lambda^{-1})$, and $\mathbf{n} = (p, p)$. Then $\mathbf{c}^{\mathbf{n-1}} = c^{2(p-1)} \neq 1$, since $p \neq 2$. □

We conclude that there exists precisely one isomorphism type of non-semisimple pointed Hopf algebra of dimension p^3 which is symmetric as a coalgebra, namely $H(\lambda, p-1)$.

3.4. **Pointed Hopf algebras of dimension 16.** The classification of pointed Hopf algebras of dimension 16 was done in [5]. There exist 43 isomorphism classes of such Hopf algebras, which we group in the following classes: 14 group Hopf algebras, corresponding to the 14 groups of order 16, one Hopf algebra with coradical kC_2, namely $E(3)$ (see Subsection 3.2), 15 liftings of quantum lines, and 13 liftings of quantum planes. Except the group Hopf algebras, the only candidates to be symmetric are the liftings of quantum planes. By using the criteria for unimodularity, it is easy to discard 6 of them. The other ones are as follows.

First take c be a generator of C_4, and c^* be a generator of C_4^*. Then $c^*(c)$ is a primitive fourth root of unity. We have the following Hopf algebras of dimension 16 with coradical kC_4.

• $H(C_4, (2,2), (c^2, c^2), (c^*, c^*), \mathbf{0}, \mathbf{0})$. Then $\mathbf{c}^{\mathbf{n-1}} = c^2c^2 = 1$, so this is unimodular. We have $\mathbf{q} = (-1,-1)$, and then if $\mathbf{c}^{\mathbf{d}} = 1$, we have $c^{2d_1+2d_2} = 1$. Hence $d_1 + d_2$ is even, and then $\mathbf{q}^{\mathbf{d}} = (-1)^{d_1+d_2} = 1$. Thus this Hopf algebra is symmetric.

• $H(C_4, (2,2), (c, c^3), ((c^*)^2, (c^*)^2), (0,1), \mathbf{0})$, $H(C_4, (2,2), (c, c^3), ((c^*)^2, (c^*)^2), \mathbf{0}, \mathbf{0})$ and $H(C_4, (2,2), (c, c^3), ((c^*)^2, (c^*)^2), (1,1), \mathbf{0})$ are non-isomorphic Hopf algebras which are isomorphic as coalgebras. For any of these Hopf algebras we have $\mathbf{c}^{\mathbf{n-1}} = cc^3 = 1$, so they are unimodular. We have $\mathbf{q} = (-1,-1)$, and then if $\mathbf{c}^{\mathbf{d}} = 1$, we have $c^{d_1+3d_2} = 1$. Hence $d_1 + 3d_2$ is a multiple of 4, therefore $d_1 + d_2$ is even, and we have $\mathbf{q}^{\mathbf{d}} = (-1)^{d_1+d_2} = 1$, and these Hopf algebras are symmetric.

• $H(C_4, (2,2), (c^2, c^2), (c^*, (c^*)^3), \mathbf{0}, \mathbf{0})$. We have $\mathbf{c}^{\mathbf{n-1}} = c^2c^2 = 1$, so it is unimodular. Also $\mathbf{q} = (-1,-1)$, and then if $\mathbf{c}^{\mathbf{d}} = 1$, we have $c^{2(d_1+d_2)} = 1$, i.e. $d_1 + d_2$ is even, so $\mathbf{q}^{\mathbf{d}} = 1$. Therefore this is also symmetric.

The following Hopf algebras have coradical $k(C_2 \times C_2)$.

• $H(C_2, (2,2), (c,c), (c^*, c^*), \mathbf{0}, \mathbf{0})$. Here c denotes the generator of C_2, and c^* is the generator of C_2^*. Thus $\mathbf{q} = (-1,-1)$. We have that $\mathbf{c}^{\mathbf{n-1}} = cc = 1$. Also, if $\mathbf{c}^{\mathbf{d}} = 1$, then $d_1 + d_2$ is even, therefore $\mathbf{q}^{\mathbf{d}} = 1$. Thus this is also symmetric.

• $H(C_2 \times C_2, (2,2), (c,c), (c^*, c^*d^*), \mathbf{0}, \mathbf{0})$, where c, d denote generators of $C_2 \times C_2$, and c^*, d^* are generators of $(C_2 \times C_2)^*$ such that $c^*(c) = -1$, $c^*(d) = 1$, $d^*(c) = 1$ and $d^*(d) = -1$. In particular $\mathbf{q} = (-1,-1)$. We have that $\mathbf{q}^{\mathbf{n-1}} = cc = 1$. If $\mathbf{c}^{\mathbf{d}} = 1$, then $d_1 + d_2$ is even, so then $\mathbf{q}^{\mathbf{d}} = 1$, and this is also symmetric.

We conclude that there exist 7 isomorphism types of non-semisimple pointed Hopf algebras of dimension 16 that are symmetric as coalgebras.

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