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LINEAR OPERATOR ON p-VALENT FUNCTION OF COMPLEX ORDER

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Abstract. We introduce two novel families of meromorphic multivalent functions, by using linear operator and study some properties (inclusion properties, basic properties) of these families. We also determine the neighborhood of these subclasses.
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1. INTRODUCTION AND DEFINITIONS

Let us denote by Σ_p the class of functions $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} (n \geq p, p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \tag{1}$$

which are analytic and p-valent in the annulus $U^* = \{z : 0 < |z| < 1, z \in \mathbb{C}\} = U \setminus \{0\}$.

We define the Hadamard product of the functions $f(z) \in \Sigma_p$ given by (1) and $g \in \Sigma_p$ given

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p} \ (p \in \mathbb{N})$$

as

$$(f * g)(z) = z^{-p} + \sum_{n=1}^{\infty} a_n b_n z^{n-p}.$$

For real or complex numbers a and c ($c \neq 0, -1, -2, \ldots$), we define the function $\varphi_p(a, c; z)$ by

$$\varphi_p(a, c; z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-p}, \tag{2}$$

where $(x)_n$ denotes the Pochhammer symbol defined by

$$(x)_n = \begin{cases} x(x+1)(x+2)\cdots(x+n-1), & n = 1, 2, 3, \ldots \\ 1, & n = 0 \end{cases}$$

We define the linear operator $\mathcal{L}_p(a, c)$ on Σ_p by the convolution

$$\mathcal{L}_p(a, c) f(z) = \varphi_p(a, c; z) * f(z). \tag{3}$$

From (3) and (2), we can write

$$z(\mathcal{L}_p(a,c)f(z))' = a\mathcal{L}_p(a+1,c)f(z) - (a+p)\mathcal{L}_p(a,c)f(z). \tag{4}$$

Let f and F be two analytic functions in the unit disk U, we say that f is subordinate to F if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ such that $f = F(w(z))$. We denote by $f \prec F$ this subordination.

Let $H_{a,c}(p; A, B, b, \mu)$ denote the class of functions of the form (1) which satisfies the condition

$$p - \frac{1}{b}\left\{\frac{z(\mathcal{L}_p(a,c)f(z))'}{\mathcal{L}_p(a,c)f(z)} + p\right\} \prec p - \mu p + p\mu\frac{1+Az}{1+Bz}, \tag{5}$$

where $0 < \mu \leq 1, a \in \mathbb{R}, c \in \mathbb{R}\backslash\{0, -1, -2, -3, \cdots\}, -1 \leq B < A \leq 1, p \in \mathbb{N}, b$ non-zero complex number.

Definition. Let $N_\delta(f)$ denotes the δ-neighborhood of the function $f \in \Sigma_p$ of the form (1), that is

$$N_\delta(f) = \{h \in \Sigma_p : h(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{n(1+|B|) + p\mu|b|(A-B)}{p\mu|b|(A-B)}\right)\frac{(a)_n}{(c)_n}|a_n - b_n| \leq \delta,$$

where

$$a > 0, c > 0, -1 \leq B < A \leq 1, \delta \geq 0\}.$$

We denote by $H^+_{a,c}(p; A, B, b, \mu)$ the class of functions $f(z) \in H_{a,c}(p; A, B, b, \mu)$, and $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} |a_n| z^n \ (p \in \mathbb{N}). \tag{6}$$

Let $N^+_\delta(f)$ denotes the δ-neighborhood of the function $f \in \Sigma_p$ of the form (6), by

$$N^+_\delta(f) = \{g \in \Sigma_p : g(z) = z^{-p} + \sum_{n=1}^{\infty} |b_n| z^n\},$$

and

$$\sum_{n=p}^{\infty}\left(\frac{\mu p(1-A)|b| + n(1-B)}{p\mu|b|(A-B)}\right)\frac{(a)_{n+p}}{(c)_{n+p}}||a_n| - |b_n|| \leq \delta,$$

where $a > 0,\ c > 0,\ -1 \leq \beta < A < 1,\ 0 < \mu \leq 1, \delta \geq 0, b$ non-zero complex number.

We can re-write the condition (5) as

$$\left|\frac{z(\mathcal{L}_p(a,c)f(z))' + p\mathcal{L}_p(a,c)f(z)}{Bz(\mathcal{L}_p(a,c)f(z))' + [Bp(1-\mu b) + A\mu b]\mathcal{L}_p(a,c)f(z)}\right| < 1. \tag{7}$$

We note that the definition of linear operator $\mathcal{L}(a,c)$ was motivated by Carlson, Shaffer [1] in space of univalent functions (see also [7]).

Also we note that the concept of δ-neighborhoods $N_\delta(f)$ of analytic functions $f(z)$ was introduced by Ruscheweyh [6] and [2], but for meromorphic p-valent function studied by Liu and Srivastava [4]. In the present paper, we derive the generalization of the result by [4].

Remark 1. (1) When we consider $b=1$ and $\mu=1$, $H_{a,c}(p;A,B,b,\mu)$ reduces to the class $H_{a,c}(p;A,B)$, which was studied by [4],

(2) When we consider $a=1$, $c=1$, $b=1$, and $\mu=1$, we get the class $H^+_{1,1}(p;A,B)$, which was investigated earlier by Mogra [5].

Lemma 1. (*Jack* [3]). *Let the function $w(z)$ (non-constant) be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in U$, then $z_0w'(z_0)=\lambda w(z_0)$, when λ is a real number and $\lambda\geq 1$.*

2. SOME BASIC PROPERTIES OF THE CLASSES $H^+_{A,C}(P;A,B,B,\mu)$ AND $H_{A,C}(P;A,B,B,\mu)$

In this section we consider $a>0,\ c>0,\ A+B\leq 0$, where $(-1\leq B<A\leq 1)$.

We start to derive the necessary and sufficient condition of the function in the class $H^+_{a,c}(p;A,B,b,\mu)$.

Theorem 1. *Let a function $f(z)$ defined by* (6) *be in Σ_p. Then the function $f(z)$ belongs to the class $H^+_{a,c}(p;A,B,b,\mu)$ if and only if*

$$\sum_{n=p}^{\infty}[n(1-B)+p(1-B-\mu|b|(A-B))]\frac{(a)_{n+p}}{(c)_{n+p}}|a_n|\leq p\mu|b|(A-B). \tag{8}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^{-p}+\left(\frac{p\mu|b|(A-B)}{n(1-B)+p(1-B-\mu|b|(A-B))}\right)\cdot\frac{(c)_{k+p}}{(a)_{k+p}}z^k \tag{9}$$

$(k=p,p+1,p+2,\cdots,n\in\mathbb{N})$.

Proof. Assuming that the inequality (8) holds true then, from (8), we find that

$$\left|\frac{z(\mathcal{L}_p(a,c)f(z))'+p\mathcal{L}_p(a,c)f(z)}{Bz(\mathcal{L}_p(a,c)f(z))'+[Bp(1-\mu b)+Ap\mu b]\mathcal{L}_p(a,c)f(z)}\right|$$
$$\leq\frac{\sum_{n=p}^{\infty}(n+p)\frac{(a)_{n+p}}{(c)_{n+p}}|a_n|}{p\mu|b|(A-B)+\sum_{n=p}^{\infty}[B(n+p)+p\mu|b|(A-B)]\frac{(a)_{n+p}}{(c)_{n+p}}|a_n|}<1$$
$(z\in U,z\in\mathbb{C},|z|=1)$.

Hence, by the Maximum Modulus Theorem we have $f(z)\in H^+_{a,c}(p;A,B,b,\mu)$.

Conversely, suppose that $f(z)$ is in the class $H^+_{a,c}(p;A,B,b,\mu)$ with $f(z)$ of the form (6), then we find, from (7), that

$$\left|\frac{z(\mathcal{L}_p(a,c)f(z))' + p\mathcal{L}_p(a,c)f(z)}{Bz(\mathcal{L}_p(a,c)f(z))' + [Bp(1-\mu b) + Ap\mu b]\mathcal{L}_p(a,c)f(z)}\right| =$$

$$\left|\frac{\sum_{n=p}^{\infty} \frac{(a)_{n+p}}{(c)_{n+p}}[n+p]|a_n|z^{n+p}}{p\mu|b|(A-B) + \sum_{n=p}^{\infty}[B(n+p) + p\mu b(A-B)]\frac{(a)_{n+p}}{(c)_{n+p}}|a_n|z^{n+p}}\right| < 1.$$

If we choose z to be real and $z \to 1^-$, we get

$$\sum_{n=p}^{\infty} \frac{(a)_{n+p}}{(c)_{n+p}}(n+p)|a_n| \le p\mu|b|(A-B) + \sum_{n=p}^{\infty}[B(n+p) + p\mu|b|(A-B)]\frac{(a)_{n+p}}{(c)_{n+p}}|a_n|$$

which is precisely the assertion (8) of Theorem 1.

Finally, we note that the assertion (8) of Theorem 1 is sharp, the extremal function being given by (9). This completes the proof of Theorem 1. □

Next, we derive the sufficient condition of function in the class

$$H_{a,c}(p;A,B,b,\mu)$$

in the next theorem also we omit the proof as in same line of proof of Theorem 1.

THEOREM 2. *Let $f \in \Sigma_p$ be given by (1). Then the sufficient condition for $f(z)$ be in the class $H_{a,c}(p;A,B,b,\mu)$, that satisfies the condition*

$$\sum_{n=1}^{\infty}[n(1-B) - p\mu|b|(A-B)]\frac{(a)_n}{(c)_n} \le p\mu|b|(A-B)$$

THEOREM 3. *If $f(z) \in H^+_{a,c}(p;A,B,b,\mu)$, then*

$$|f^{(m)}(z)| \le \left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(c)_{2p}}{(a)_{2p}}\left(\frac{\mu|b|(A-B)}{2(1-B)-\mu(|b|(A-B)}\right)\frac{p!}{(p-m)!}r^{2p}\right\}r^{-p-m}, \tag{10}$$

$$|f^{(m)}(z)| \ge \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{(c)_{2p}}{(a)_{2p}}\left(\frac{\mu|b|(A-B)}{2(1-B)-\mu(|b|(A-B)}\right)\frac{p!}{(p-m)!}r^{2p}\right\}r^{-p-m} \tag{11}$$

($0 < |z| = r < 1, a > c > 0, m \in \mathbb{N}_0, p > m, 0 < \mu \le 1, -1 \le B < A \le 1, b$ complex number.)

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \left(\frac{\mu|b|(A-B)}{2(1-B) - \mu|b|(A-B)} \right) \frac{(c)_{2p}}{(a)_{2p}} z^p \ (p \in \mathbb{N}). \tag{12}$$

Proof. Suppose that $f(z) \in H^+_{a,c}(p; A, B, b, \mu)$, then we find, from (8), that

$$\frac{(a)_{2p}}{(c)_{2p}} \cdot \frac{p[2(1-B) - \mu|b|(A-B)]}{p!} \sum_{n=p}^{\infty} n!|a_n|$$

$$\leq \sum_{n=p}^{\infty} [n(1-B) + p(1-B-\mu|b|(A-B))] \frac{(a)_{n+p}}{(c)_{n+p}} |a_n| \leq p\mu|b|(A-B).$$

We conclude that

$$\sum_{n=p}^{\infty} n!|a_n| \leq \left(\frac{\mu|b|(A-B)}{2(1-B) - \mu|b|(A-B)} \right) \frac{(c)_{2p}}{(a)_{2p}} p!. \tag{13}$$

If we differentiate both sides of (6) m times with respect to z, we get

$$f^{(m)}(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-p-m} + \sum_{n=p}^{\infty} \frac{n!}{(n-m)!} |a_n| z^{n-m}$$

$$(m \in \mathbb{N}_0, p \in \mathbb{N}, p > m)$$

$$= (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-p-m} + \sum_{n=p}^{\infty} n! \varphi(n) |a_n| z^{n-m}, \tag{14}$$

where, for convenience,

$$\varphi(n) = \frac{1}{(n-m)!} \ (p \in \mathbb{N}, m \in \mathbb{N}_0, n \geq p, m < p).$$

Clearly, the function $\varphi(n)$ is decreasing in n, and we have

$$0 < \varphi(n) \leq \varphi(p) = \frac{1}{(p-m)!} \tag{15}$$

Making use of (13), (14) and (15), we get (10) and (11).

In order to complete the proof of Theorem 3, it is easily observed the equalities in (10) and (11) are satisfied by the function $f(z)$ given by (12). □

THEOREM 4. *Let the function $f(z)$ defined by (6) be in the class*

$$H^+_{a,c}(p; A, B, b, \mu).$$

Then

(i) *f is meromorphically p-valent starlike of order $\delta(0 \leq \delta < p)$ in the disk $|z| < r_1$, where*

$$r_1 = r_1(p; A, B, b, \mu; \delta)$$

$$= \inf_{n \geq p} \left(\frac{(a)_{n+p}}{(c)_{n+p}} \cdot \frac{n(1-B) + (1-B-\mu|b|(A-B))}{\mu|b|(A-B)} \cdot \frac{(p-\delta)}{n+\delta} \right)^{\frac{1}{n+p}}. \tag{16}$$

(ii) *f is meromorphically p-valent convex of order $\delta(0 \leq \delta < p)$ in the disk $|z| < r_2$, where*

$$r_2 = r_2(p; A, B, b, \mu; \delta) = \inf_{n\geq p}\left(\frac{(a)_{n+p}}{(c)_{n+p}} \cdot \frac{p(p-\delta)(n(1-B)+(1-B-\mu|b|(A-B))}{n(n+\delta)\mu|b|(A-B)}\right)^{\frac{1}{n+p}}.$$

Proof. (i) Making use of the definition (6), it is not difficult to observe that

$$\left|\frac{zf'(z)+pf(z)}{zf'(z)+(2\delta-p)f(z)}\right| \leq \frac{\sum_{n=p}^{\infty}(n+p)|a_n||z|^{n+p}}{2(p-\delta)-\sum_{n=p}^{\infty}(n+p+2\delta)|a_n||z|^{n+p}} \leq 1 \tag{17}$$

$(|z| < r_1; 0 \leq \delta < 1)$.

This last inequality (17) holds true if

$$\sum_{n=p}^{\infty}\left(\frac{n+\delta}{p-\delta}\right)|a_n||z|^{n+p} \leq 1.$$

In view of (8), the last inequality is true if

$$\left(\frac{n+\delta}{p-\delta}\right)|z|^{n+p} \leq \left(\frac{n(1-B)+p(1-B-\mu|b|(A-B))}{\mu|b|(A-B)}\right)\frac{(a)_{n+p}}{(c)_{n+p}}$$
$$(n \geq p, p \in \mathbb{N})$$

which, when solved for $|z|$, yields (16).

(ii) Making use of the definition (6), it is easy to observe that

$$\left|\frac{zf''(z)+(1+p)f'(z)}{zf''(z)+(1-p+2\delta)f'(z)}\right| \leq \frac{\sum_{n=p}^{\infty}n(n+p)|a_n||z|^{n+p}}{2p(p-\delta)-\sum_{n=p}^{\infty}n(n-p+2\delta)|a_n||z|^{n+p}} \leq 1 \tag{18}$$

$(z < r_2, 0 \leq \delta < 1)$.

The last inequality (18) holds true if

$$\sum_{n=p}^{\infty}\left(\frac{n(n+\delta)}{p(p-\delta)}\right)|a_n|\cdot|z|^{n+p} \leq 1. \tag{19}$$

According to Theorem 3, the inequality (19) is true if

$$\left(\frac{n(n+\delta)}{p(p-\delta)}\right)|z|^{n+p} \leq \left(\frac{n(1-B)+p(1-B-\mu|b|(A-B)}{\mu|b|(A-B))}\right)\cdot\frac{(a)_{n+p}}{(c)_{n+p}}$$
$$(n \geq p, p \in \mathbb{N})$$

or if

$$|z| \leq \left(\frac{(a)_{n+p}}{(c)_{n+p}} \cdot \frac{n(1-B)+p(1-B-\mu|b|(A-B))}{\mu|b|(A-B)}\frac{p(p-\delta)}{n(n+\delta)}\right)^{\frac{1}{n+p}}. \tag{20}$$

The theorem follows easily from (20). □

3. INCLUSION PROPERTIES OF THE CLASS $H_{A,C}(P; A, B, B, \mu)$

In this section we make use of the Jack's Lemma to prove the theorems in this section.

THEOREM 5. *Let* $a \geq \frac{p\mu|b|(A-B)}{1+B}$, *then we have*

$$H_{a+1,c}(p; A, B, b, \mu) \subset H_{a,c}(p; A, B, b, \mu),$$

where $-1 \leq B < A \leq 1$, $0 < \mu \leq 1$, $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$, $p \in \mathbb{N}$, b *a (non-zero) complex number.*

Proof. Suppose $f \in H_{a+1,c}(p; A, B, b, \mu)$, and

$$\frac{z(\mathcal{L}_p(a,c)f(z))'}{\mathcal{L}_p(a,c)f(z)} = -p\left(1 - \mu|b| + \mu|b| \cdot \frac{1 + Aw(z)}{1 + Bw(z)}\right) \tag{21}$$

for some either analytic or meromorphic function $w(z)$ in U, with $w(0) = 0$.
From (21) and (4) we get

$$\frac{a\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} = \frac{a + [aB - p\mu|b|(A-B)]w(z)}{1 + Bw(z)} \tag{22}$$

differentiating logarithmically both sides of (22) we get

$$\frac{z(\mathcal{L}_p(a+1,c)f(z))'}{\mathcal{L}_p(a+1,c)f(z)} = \frac{z(\mathcal{L}_p(a,c)f(z))'}{\mathcal{L}_p(a,c)f(z)} + \frac{(aB - p\mu|b|(A-B))zw'(z)}{a + [aB - p\mu|b|(A-B)]w(z)}$$
$$-\frac{Bz'w(z)}{1 + Bw(z)} = -p\left[\frac{1 + (B + \mu|b|(A-B))w(z)}{1 + Bw(z)}\right]$$
$$-\frac{p\mu|b|(A-B)zw'(z)}{(1 + Bw(z))[a + (aB - p\mu|b|(A-B))w(z)]}. \tag{23}$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1, \ (w(z_0) \neq -1),$$

then Jack's Lemma gives us that

$$z_0 w'(z_0) = kw(z_0), \ (k \geq 1).$$

Now setting $w(z_0) = \mathrm{e}^{\mathrm{i}\theta} (\theta \neq \pi)$ in (23) we get

$$\left|\frac{z_0(\mathcal{L}_p(a,c)f(z_0))' + p\mathcal{L}_p(a,c)f(z_0)}{Bz_0(\mathcal{L}_p(a,c)f(z_0))' + [Bp(1 - \mu b) + Ap\mu b]\mathcal{L}_p(a,c)f(z_0)}\right|^2 - 1$$
$$= \left|\frac{-p(a+k) + [aB - p\mu b(A-B)]\mathrm{e}^{\mathrm{i}\theta}}{a + [aB - kB - p\mu b(A-B)]\mathrm{e}^{\mathrm{i}\theta}}\right|^2 - 1$$
$$\geq \left|\frac{(a+k) + [aB - p\mu b(A-B)]\mathrm{e}^{\mathrm{i}\theta}}{a + [aB - k - p\mu b(A-B)]\mathrm{e}^{\mathrm{i}\theta}}\right|^2 - 1$$

But we have contradiction with the condition of the theorem.
Therefore we have $|w(z)| < 1$, and by (21) we get $f \in H_{a,c}(p; A, B, b, \mu)$.

This completes the proof. □

THEOREM 6. *Let the function $f(z)$ defined by (1) be in the class*

$$H_{a,c}(p; A, B, b, \mu).$$

Then the function $g(z)$ defined by

$$\mathcal{L}_p(a,c)g(z) = \left(\frac{\lambda - p\alpha}{z^\lambda}\right) \int_0^z t^{\lambda-1}[\mathcal{L}_p(a,c)f(t)]^\alpha \mathrm{d}t]^{1/\alpha} \tag{24}$$

(where $\alpha > 0, \lambda > p\alpha \cdot \left(\frac{1+[B+\mu|b|(A-B)]}{1+B}\right) > 0, p \in \mathbb{N}$) is also in same class $H_{a,c}(p; A, B, b, \mu)$.

Proof. Let $f \in H_{a,c}(p; A, B, b, \mu)$, and

$$\frac{z(\mathcal{L}_p(a,c)g(z))'}{\mathcal{L}_p(a,c)g(z)} = -p\left(1 - \mu|b| + \mu|b| \cdot \frac{1 + Aw(z)}{1 + Bw(z)}\right) \tag{25}$$

for some either analytic or meromorphic function $w(z)$ in U, with $w(0) = 0$.
From (6), we have

$$[\mathcal{L}_p(a,c)g(z)]^\alpha = \frac{\lambda - p\alpha}{z^\lambda} \int_0^z t^{\lambda-1}[\mathcal{L}_p(a,c)f(t)]^\alpha \mathrm{d}t. \tag{26}$$

Differentiating logarithmically both sides of (26), and after some computation, we get

$$\frac{z(\mathcal{L}_p(a,c)g(z))'}{\mathcal{L}_p(a,c)g(z)} = -\frac{\lambda}{\alpha} + \frac{\lambda - p\alpha}{\alpha}\left[\frac{\mathcal{L}_p(a,c)f(z)}{\mathcal{L}_p(a,c)g(z)}\right]^\alpha \tag{27}$$

then from (25) and (27), we have

$$\frac{\lambda(\mathcal{L}_p(a,c)f(z))^\alpha + (\alpha p - \lambda)(\mathcal{L}_p(a,c)f(z))^\alpha}{(\mathcal{L}_p(a,c)g(z))^\alpha} = \alpha p\left[\frac{1 + (B + \mu|b|(A-B))w(z)}{1 + Bw(z)}\right].$$

Differentiating both sides of last equality, and from (25) and (27), we get

$$\frac{z(\mathcal{L}_p(a,c)f(z))'}{\mathcal{L}_p(a,c)f(z)} = \frac{p(1 + [B + \mu|b|(A-B)]w(z))}{\alpha p(1 + [B + \mu|b|(A-B)]w(z) - \lambda(1 + Bw(z))} \times \left[\lambda - \alpha p\left\{\frac{(1 + [B + \mu|b|(A-B)]w(z) + Bzw'(z)}{1 + Bw(z)}\right\} + \frac{[B + \mu|b|(A-B)]zw'(z)}{1 + [B + \mu|b|(A-B)]w(z)}\right]. \tag{28}$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1, \ (w(z_0) \neq -1),$$

then by Jack's Lemma $z_0w'(z_0) = kw(z_0)$, $(k \geq 1)$. Now letting $w(z_0) = \mathrm{e}^{\mathrm{i}\theta}(\theta \neq \pi)$, in (28), we get

$$\left|\frac{z_0(\mathcal{L}_p(a,c)f(z_0))' + p\mathcal{L}_p(a,c)f(z_0)}{Bz_0(\mathcal{L}_p(a,c)f(z_0))' + [Bp(1-\mu b) + pA\mu b]\mathcal{L}_p(a,c)f(z_0)}\right|^2 - 1$$

$$= \left|\frac{\lambda + k - \alpha p + [B\lambda - \alpha p(B + \mu b(A-B))]\mathrm{e}^{\mathrm{i}\theta}}{\lambda - \alpha p + [B\lambda - Bk - \alpha p[B + \mu b(A-B)]\mathrm{e}^{\mathrm{i}\theta}}\right|^2 - 1$$

$$= \frac{q(\theta)}{|\lambda - \alpha p + (B\lambda - Bk - \alpha p(B + \mu|b|(A-B)\mathrm{e}^{\mathrm{i}\theta}|^2}, \tag{29}$$

where

$$\begin{aligned} q(\theta) &= k^2(1-B^2) + 2k[(1+B^2)\lambda - \alpha p(1 + B(B + \mu|b|(A-B)))] \\ &\quad +2k[2B\lambda - \alpha p(2B + \mu|b|(A-B))\cos\theta, \end{aligned}$$

where $0 \leq \theta < 2\pi$, $-1 \leq B < A \leq 1$, $k \geq 1$, $0 < \mu \leq 1$, b a complex number, by hypothesis we have $\lambda \geq p\alpha\frac{1+[B+\mu|b|(A-B)]}{1+B} > p\alpha\frac{1-[B+\mu|b|(A-B)]}{1-B}$. Therefore

$$q(\theta) \geq 0 \text{ and } q(\pi) \geq 0. \tag{30}$$

Hence, from (30), we obtain

$$q(\theta) \geq 0 \ (0 \leq \theta < 2\pi). \tag{31}$$

In view of (31) and (29), we get contradiction with the condition of the theorem that $f \in H_{a,c}(p, A, B, b, \mu)$. Therefore we have $|w(z)| < 1$, and by (21) we get $g(z) \in H_{a,c}(p; A, B, b, \mu)$. This completes the proof. □

4. NEIGHBORHOOD OF THE CLASSES $H_{A,C}(P; A, B, B, \mu)$ AND $H^{+}_{A,C}(P; A, B, B, \mu)$

We start to prove the neighborhood of the class $H_{a,c}(p; A, B, b, \mu)$.

THEOREM 7. *Let the function $f(z)$ defined by* (1) *be in $H_{a,c}(p; A, B, b, \mu)$. For all ϵ in $\mathbb{C}$ with $|\epsilon| < \delta$, let*

$$\frac{f(z) + \epsilon z^{-p}}{1+\epsilon} \in H_{a,c}(p; A, B, b, \mu),$$

then

$$N_\delta(f) \subset H_{a,c}(p; A, B, b, \mu) \ (\delta > 0).$$

Proof. Let $g \in H_{a,c}(p; A, B, b, \mu)$, then from (1.7), we can write, for $(\gamma \in \mathbb{C}, |\gamma| = 1)$, that

$$\left[\frac{z(\mathcal{L}_p(a,c)g(z))' + p\mathcal{L}_p(a,c)g(z)}{Bz(\mathcal{L}_p(a,c)g(z))' + [Bp(1-\mu|b|) + Ap\mu|b|]\mathcal{L}_p(a,c)g(z)}\right] \neq \gamma. \tag{32}$$

Let

$$h(z) = z^{-p} + \sum_{n=1}^{\infty} c_n z^{n-p}.$$

In view of (32), we can write

$$h(z) = z^{-p} + \sum_{n=1}^{\infty} \left(\frac{n(1-\gamma B) - p\gamma\mu|b|(A-B)}{\gamma p\mu|b|(A-B)} \right) \frac{(a)_n z^{n-p}}{(c)_n}. \tag{33}$$

From (33), we obtain

$$|c_n| = \left| \left(\frac{n(1-\gamma B) - p\gamma\mu|b|(A-B)}{\gamma p\mu|b|(A-B)} \right) \frac{(a)_n}{(c)_n} \right| \leq \frac{\mu|b|p(A-B) + n(1+|B|)}{p\mu|b|(A-B)}$$

where $n \geq p,\ p \in \mathbb{N}$.

Now we can write (32) as

$$\frac{(g*h)(z)}{z^{-p}} \neq 0 \ (z \in U). \tag{34}$$

By the condition of the theorem, (34) reduced to

$$\frac{1}{z^{-p}} \cdot \frac{(f*h)(z) + \epsilon z^{-p}}{1+\epsilon} \neq 0, \text{ or } \left| \frac{(f*g)(z)}{z^{-p}} \right| \geq \delta.$$

Let

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p} \in N_\delta(f),$$

then

$$\left| \frac{(g*h)}{z^{-p}} \right| = \left| \frac{f*h}{z^{-p}} + \frac{(g-f)*h}{z^{-p}} \right| \geq \delta - \left| \frac{(g-f)*h}{z^{-p}} \right|.$$

But we have

$$\left| \frac{(g-f)*h}{z^{-p}} \right| = \left| \sum_{n=1}^{\infty} (a_n - b_n) c_n z^n \right|$$

$$\leq |z| \sum_{n=1}^{\infty} \left(\frac{p\mu|b|(A-B) + n(1+|B|)}{p\mu|b|(A-B)} \right) \frac{(a)_n}{(c)_n} |a_n - b_n| \leq \delta \ (z \in U, \delta > 0).$$

From (34) and (32), we get $g \in H_{a,c}(p; A, B, b, \mu)$, so $N_\delta \subset H_{a,c}(p; A, B, b, \mu)$. This completes the proof. □

THEOREM 8. *Let $f(z)$ be defined by* (1)*, and let the partial sums $S_1(z)$ and $S_k(z)$ are defined by $S_1(z) = z^{-p}$ and*

$$S_k(z) = z^{-p} + \sum_{n=1}^{k-1} a_n z^{n-p} \ (k \in \mathbb{N}, k > 1).$$

Also suppose that

$$\sum_{n=1}^{\infty} \left(\frac{p\mu|b|(A-B) + n(1+|B|)}{p\mu|b|(A-B)} \right) \frac{(a)_n}{(c)_n} \leq 1. \tag{35}$$

If $a > 0$ *and* $c > 0$*, then* $f(z) \in H_{a,c}(p; A, B, b, \mu)$*, and if* $a > c > 0$*, then*

$$R\left(\frac{f(z)}{S_k(z)}\right) > 1 - \frac{1}{d_k} \ (z \in U, \ k \in \mathbb{N}), \tag{36}$$

and

$$R\left(\frac{S_k(z)}{f(z)}\right) > \frac{d_k}{1 + d_k} \ (z \in U, k \in \mathbb{N}). \tag{37}$$

Each of (36) *and* (37) *is the best possible for* $kn \in \mathbb{N}$*.*

Proof. By applying Theorem 7 and from (35), we get

$$N_1(z^{-p}) \in H_{a,c}(p; A, B, b, \mu) \ (a > 0, \ c > 0, \ p \in \mathbb{N}),$$

where $z^{-p} \in H_{a,c}(p; A, B, b, \mu)$ (from (7)). Thus $f \in H_{a,c}(p; A, B, b, \mu)$.

Now, from (35), we find that

$$\sum_{n=1}^{k-1} |a_n| + d_k \sum_{n=k}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} d_n |a_n| \leq 1 \ (\text{as } d_{n+1} > d_n > 1). \tag{38}$$

Setting

$$\begin{aligned} h_1(z) &= d_k \left\{ \frac{f(z)}{S_k(z)} - \left(1 - \frac{1}{d_n}\right) \right\} \\ &= d_k \left\{ \frac{z^{-p} + \sum_{n=k}^{\infty} a_n z^{n-p}}{z^{-p} + \sum_{n=1}^{k-1} a_n z^{n-p}} - 1 + \frac{1}{d_k} \right\} = \frac{d_k \sum_{n=k}^{\infty} a_n z^n}{1 + \sum_{n=1}^{k-1} a_n z^n} + 1, \end{aligned}$$

from (38), we get

$$\begin{aligned} \left| \frac{h_1(z) - 1}{h_1(z) + 1} \right| &= \frac{d_k \sum_{n=k}^{\infty} a_n z^n}{2 + 2\sum_{n=1}^{k-1} a_n z^n + d_k \sum_{n=k}^{\infty} a_n z^n} \\ &\leq \frac{d_k \sum_{n=k}^{\infty} |a_n|}{2 - 2\sum_{n=1}^{k-1} |a_n| - d_k \sum_{n=k}^{\infty} |a_n|} \leq \frac{d_k \sum_{n=1}^{\infty} |a_n|}{1 - \sum_{n=1}^{k-1} |a_n|} \leq 1. \end{aligned}$$

This proves (36).

To prove that the bound in (36) is the best possible for each $k \in \mathbb{N}$, set $f(z) = z^{-p} - \frac{z^{k-p}}{d_k}$. Then

$$\frac{f(z)}{S_k(z)} = 1 - \frac{z^k}{d_k}. \tag{39}$$

When $z \to 1$, (39) reduces to $(1 - 1/d_k)$, which proves the assertion of the theorem.

Now, in the same way we can prove (37), by setting

$$g(z) = (1 + d_k)\left(\frac{S_k(z)}{f(z)} - \frac{d_k}{1 + d_k}\right).$$

This completes the proof. □

THEOREM 9. *Let the function $f(z)$ defined by (6) be in the class*

$$H^+_{a,c}(p; A, B, b, \mu),$$

and $A + B \leq 0$, then

$$N^+_\delta(f) \subset H^+_{a,c}(p; A, B, b, \mu),$$

where $\delta := \frac{2p}{a+2p}$.

The result is sharp.

Proof. We can prove this theorem in the same way as Theorem 7, with

$$\begin{aligned} h(z) &= z^{-p} + \sum_{n=p}^{\infty} c_n z^n \\ &= z^{-p} + \sum_{n=p}^{\infty} \left[\frac{(n+p)(1-\gamma B) - \gamma p \mu |b| (A-B))}{\mu |b| p \gamma (B-A)} \frac{(a)_{n+p}}{(c)_{n+p}} z^n \right], \end{aligned}$$

and for $A + B \leq 0$ and $f \in H^+_{a,c}(p; A, B, b, \mu)$, we have

$$\left| \frac{(f * h)(z)}{z^{-p}} \right| \geq \frac{2p}{a+2p} = \delta.$$

Next, to prove the sharpness, let $f(z)$ and $g(z)$ given by

$$f(z) = z^{-p} + \left(\frac{\mu |b| (A-B)}{2 - 2B - \mu |b| (A-B)} \right) \frac{(c)_{2p}}{(a+1)_{2p}} z^p \in H^+_{a,c}(p; A, B, b, \mu)$$

and $g(z) =$

$$z^{-p} + \left[\frac{\mu |b| (A-B)}{2 - 2B - \mu |b| (A-B)} \cdot \frac{(c)_{2p}}{(a+1)_{2p}} + \frac{\mu |b| \delta' (A-B)}{2 - 2B - \mu |b| (A-B)} \cdot \frac{(c)_{2p}}{(a)_{2p}} \right] z^p,$$

where $\left(\delta' > \delta = \frac{2p}{a+2p} \right)$, we get $g(z) \in N^+_\delta(f)$ but $g(z) \notin H^+_{a,c}(p; A, B, b, \mu)$. □

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