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NEW CRITERIA FOR MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Let $K_n(\alpha)$ be the class of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k \qquad (a_{-1} \neq 0)$$

which are regular in the punctured disc $\mathrm{U}^* = \{z : 0 < |z| < 1\}$ and satisfy

$$\mathrm{Re}\left\{-z^2 \left(D^n f(z)\right)'\right\} > \alpha,\ 0 \le \alpha < 1,\ |z| < 1,$$

and $n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$, where

$$D^n f(z) = \frac{a_{-1}}{z} + \sum_{k=2}^{\infty} k^n a_{k-2} z^{k-2}.$$

It is proved that $K_{n+1}(\alpha) \subset K_n(\alpha)$. Since $K_0(\alpha)$ is the class of meromorphically close-to-convex functions, all functions in $K_n(\alpha)$ are meromorphically close-to-convex.

MSC 2000. 30C45.
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1. INTRODUCTION

Let $\sum$ denote the class of functions of the form:

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k, \quad (a_{-1} \neq 0) \tag{1.1}$$

which are regular in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Define

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= \frac{a_{-1}}{z} + 2a_0 + 3a_1 z + 4a_2 z^2 + \dots \\ &= \frac{\left(z^2 f(z)\right)'}{z}, \\ D^2 f(z) &= D^1\left(D^1 f(z)\right), \end{aligned}$$

and for $n = 1, 2, 3, \dots$

$$D^n f(z) = D^1\left(D^{n-1} f(z)\right) = \frac{a_{-1}}{z} + \sum_{k=2}^{\infty} k^n a_{k-2} z^{k-2}. \tag{1.2}$$

Let $K_n(\alpha)$ denote the class of functions $f(z)$ which satisfy the condition

$$\text{(1.3)} \qquad \operatorname{Re}\left\{-z^2\left(D^n f(z)\right)'\right\} > \alpha,$$

$0 \le \alpha < 1,\ |z| < 1,\ n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $D^n f(z)$ is defined by (1.2).

In this paper we shall show that

$$\text{(1.4)} \qquad K_{n+1}(\alpha) \subset K_n(\alpha), \quad 0 \le \alpha < 1, \quad n \in \mathbb{N}_0.$$

Since $K_0(\alpha)$ is the class of functions $f(z) \in \sum$ which satisfy $\operatorname{Re}\left\{-z^2 f'(z)\right\} > \alpha$ for $|z| < 1$, it follows from (1.4) that all functions in $K_n(\alpha)$ are meromorphically close-to-convex. Further we consider the integrals of functions in $K_n(\alpha)$.

In [3] Uralegaddi and Somanatha obtain a new criterion for meromorphic starlike functions via the basic inclusion relationship $B_{n+1}(\alpha) \subset B_n(\alpha)$, $0 \le \alpha < 1$ and $n \in \mathbb{N}_0$, where $B_n(\alpha)$ is the class of functions $f(z) \in \sum$ satisfying

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - 2\right\} < -\alpha,$$

$0 \le \alpha < 1,\ n \in \mathbb{N}_0$ and $|z| < 1$.

2. PROPERTIES OF THE CLASS $K_N(\alpha)$

In proving our main results (Theorem 1 and Theorem 2 below), we shall need the following lemma due to Jack [2].

LEMMA 1. *Let $w(z)$ be non-constant regular in $U = \{z : |z| < 1\}$, $w(0) = 0$. If $w(z)$ attains its maximum value on the circle $|z| = r < 1$ at z_o, we have $w'(z_o) = k w(z_o)$, where k is a real number, $k \ge 1$.*

THEOREM 1. *$K_{n+1}(\alpha) \subset K_n(\alpha)$ for each $n \in N_o$.*

Proof. Let $f(z) \in K_{n+1}(\alpha)$. Then

$$\text{(2.1)} \qquad \operatorname{Re}\left\{-z^2\left(D^{n+1} f(z)\right)'\right\} > \alpha, \quad |z| < 1.$$

We have to show that (2.1) implies the inequality

$$\text{(2.2)} \qquad \operatorname{Re}\left\{-z^2\left(D^n f(z)\right)'\right\} > \alpha.$$

Define a regular function $w(z)$ in the unit disc $U = \{z : |z| < 1\}$ by

$$\text{(2.3)} \qquad -z^2\left(D^n f(z)\right)' = \frac{1 + (2\alpha - 1) w(z)}{1 + w(z)}.$$

Differentiating (2.3) we obtain

$$\text{(2.4)} \qquad z^2\left(D^n f(z)\right)'' + 2z\left(D^n f(z)\right)' = \frac{2(1-\alpha) w'(z)}{(1 + w(z))^2}.$$

One can easily verify the identity

$$\text{(2.5)} \qquad z\left(D^n f(z)\right)' = D^{n+1} f(z) - 2D^n f(z).$$

Differentiating (2.5) we obtain

$$z^2 \left(D^n f(z)\right)'' = z \left(D^{n+1} f(z)\right)' - 3z \left(D^n f(z)\right)'. \tag{2.6}$$

Using (2.6), (2.4) may be written as

$$-z^2 \left(D^{n+1} f(z)\right)' = -z^2 \left(D^n f(z)\right)' - \frac{2(1-\alpha) z w'(z)}{(1+w(z))^2}. \tag{2.7}$$

We claim that $|w(z)| < 1$ in U. For otherwise (by Jack's lemma 1) there exists a point z_o in U such that

$$z_o w'(z_o) = k w(z_o), \tag{2.8}$$

where $|w(z_o)| = 1$ and $k \geq 1$. From (2.7) and (2.8), we obtain

$$-z_o^2 \left(D^{n+1} f(z_o)\right)' = \frac{1 + (2\alpha - 1) w(z_o)}{1 + w(z_o)} - \frac{2(1-\alpha) k w(z_o)}{(1+w(z_o))^2}.$$

Thus

$$\operatorname{Re}\left\{-z_o^2 \left(D^{n+1} f(z_o)\right)'\right\} = \alpha - 2(1-\alpha) k \operatorname{Re} \frac{w(z_o)}{(1+w(z_o))^2} \leq \alpha$$

which contradicts (2.1). Hence $|w(z)| < 1$ in U and from (2.3) it follows that $f(z) \in K_n(\alpha)$. □

Theorem 2. *Let* $f(z) \in K_n(\alpha)$ *and* $\operatorname{Re} c > 0$. *Then*

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\, \mathrm{d}t \in K_n(\alpha).$$

Proof. From the hypothesis we have

$$z \left(D^n F(z)\right)' + (c+1) D^n F(z) = c D^n f(z). \tag{2.9}$$

Differentiating (2.9) we obtain

$$z \left(D^n F(z)\right)'' + (c+2) \left(D^n F(z)\right)' = c \left(D^n f(z)\right)'. \tag{2.10}$$

Define $w(z)$ in U by

$$-z^2 \left(D^n F(z)\right)' = \frac{1 + (2\alpha - 1) w(z)}{1 + w(z)}. \tag{2.11}$$

Clearly $w(z)$ is regular and $w(0) = 0$. Differentiating (2.11) we obtain

$$z^2 \left(D^n F(z)\right)'' + 2z \left(D^n F(z)\right)' = \frac{2(1-\alpha) z w'(z)}{c(1+w(z))^2}. \tag{2.12}$$

Using (2.12), (2.10) may be written as

$$-z^2 \left(D^n f(z)\right)' = -z^2 \left(D^n F(z)\right)' - \frac{2(1-\alpha) z w'(z)}{c(1+w(z))^2}. \tag{2.13}$$

The remaining part of the proof is similar to that of Theorem 1. □

THEOREM 3. *Let $f(z) \in \sum$ and satisfy the condition* $\operatorname{Re}\left\{-z^2 \left(D^n f(z)\right)'\right\} > \alpha - \frac{1-\alpha}{2c}$ *where c is any real number greater than zero. Then*

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\,\mathrm{d}t \in K_n(\alpha).$$

Proof. The proof is similar to the proof of Theorem 2. □

Taking $n = \alpha = 0$ and $c = 1$, we get

COROLLARY 1. *If* $\operatorname{Re}\left\{-z^2 f'(z)\right\} > -\frac{1}{2}$ *for* $|z| < 1$, *then* $\operatorname{Re}\left\{-z^2 F'(z)\right\} > 0$ *for* $|z| < 1$, *where*

$$F(z) = \frac{1}{z^2} \int_0^z t f(t)\,\mathrm{d}t.$$

THEOREM 4. *Let* $F(z) = \frac{c}{z^{c+1}} \int_0^z \xi^c f(\xi)\,\mathrm{d}\xi$, $\operatorname{Re}(c) = t > 0$ *and* $F(z) \in K_n(\alpha)$. *Then* $\operatorname{Re}\left\{-z^2 \left(D^n f(z)\right)'\right\} > \alpha$ *for* $|z| < R_c$, *where* $R_c = \frac{\sqrt{1+t^2}-1}{t}$. *The estimate is sharp when c is real for the function $f(z)$ for which*

$$-z^2 \left(D^n f(z)\right)' = \alpha + (1-\alpha)\frac{1-z}{1+z}.$$

Proof. The proof is similar to the proof of [1, Theorem 4]. □

REFERENCES

[1] GANIGI, M.D. and URALEGADDI, B.A., *Subclasses of meromorphic close-to-convex functions*, Bull. Math. Soc. Sci. Math. R.S. Roumanie (N-S.), **33 (81)** (1989), 105–109.

[2] JACK, I.S., *Functions starlike and convex of order α*, J. London Math. Soc., **2** (1971), 469–474.

[3] URALEGADDI, B.A. and SOMANATHA, C., *New criteria for meromorphic starlike univalent functions*, Bull. Austral. Math. Soc., **43** (1991), 137–140.

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