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ON SOME PROPERTIES OF UNIVALENT FUNCTIONS IN THE UPPER HALF PLANE

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Abstract. In this paper we obtain some inequalities concerning the modulus of univalent functions in the upper half-plane.

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1. INTRODUCTION

Let D denote the upper half-plane, i.e.

$$D = \{z \in \mathbb{C} : \text{ Im } z > 0\}$$

and let $H(D)$ denote the class of all analytic functions in D.

The following inequality has been obtained by N.N. Pascu and we will use it to prove our results.

THEOREM 1. [1] *Let $f \in H(D)$ be an univalent function in D. Then*

$$\left| \mathrm{i} - \text{Im } z \frac{f''(z)}{f'(z)} \right| \leq 2, \text{ for all } z \in D. \tag{1}$$

As a consequence of Theorem 1 we have

THEOREM 2. *Let $f \in H(D)$ be an univalent function in D such that $f(z) \neq 0$, for all $z \in D$. Then*

$$(\text{Im } z) \left| \frac{f'(z)}{f(z)} \right| \leq 2, \text{ for all } z \in D. \tag{2}$$

Proof. Since $f(z) \neq 0$, for all $z \in D$, the function

$$g(z) = \frac{1}{f(z)}, \quad z \in D \tag{3}$$

is also analytic and univalent in D. It is easy to check that

$$\frac{2f'(z)}{f(z)} = \frac{f''(z)}{f'(z)} - \frac{g''(z)}{g'(z)}, \quad z \in D. \tag{4}$$

We have

$$\begin{aligned} 2(\mathrm{Im}\ z)\left|\frac{f'(z)}{f(z)}\right| &= \left|\mathrm{Im}\ z\frac{f''(z)}{f'(z)} - \mathrm{Im}\ z\frac{g''(z)}{g'(z)}\right| \\ &= \left|-\mathrm{i} + \mathrm{Im}\ z\frac{f''(z)}{f'(z)} + \mathrm{i} - \mathrm{Im}\ z\frac{g''(z)}{g'(z)}\right| \le \\ &\le \left|\mathrm{i} - \mathrm{Im}\ \frac{f''(z)}{f'(z)}\right| + \left|\mathrm{i} - \mathrm{Im}\ z\frac{g''(z)}{g'(z)}\right|, \quad z \in D. \end{aligned}$$

Since both functions f and g satisfy the inequality (1) we obtain

$$2(\mathrm{Im}\ z)\left|\frac{f'(z)}{f(z)}\right| \le 4, \quad z \in D$$

and hence inequality (2) holds true. □

2. SOME INEQUALITIES FOR THE MODULUS OF UNIVALENT FUNCTIONS IN D

If $f : D \to \mathbb{C}$ and $f(z) \neq 0$, $z \in D$ then it is easy to observe that

$$\frac{\partial}{\partial y}[\ln f(x + \mathrm{i}y)] = \mathrm{i}\frac{f'(x + \mathrm{i}y)}{f(x + \mathrm{i}y)} \tag{5}$$

and

$$\frac{\partial}{\partial x}[\ln f(x + \mathrm{i}y)] = \frac{f'(x + \mathrm{i}y)}{f(x + \mathrm{i}y)}. \tag{6}$$

Using inequalities (5), (6) and Theorem 2 we obtain

THEOREM 3. *Let $f \in H(D)$ be an univalent function in D such that $f(z) \neq 0$, for all $z \in D$. Then*

$$m(y) \le \left|\frac{f(x + \mathrm{i}y)}{f(x + \mathrm{i})}\right| \le M(y), \tag{7}$$

where

$$m(y) = \min\left\{y^2, \frac{1}{y^2}\right\}, \quad M(y) = \max\left\{y^2, \frac{1}{y^2}\right\}$$

and

$$\mathrm{e}^{-2|x|} \le \left|\frac{f(x + \mathrm{i})}{f(i)}\right| \le \mathrm{e}^{2|x|}, \tag{8}$$

for all $x \in \mathbb{R}$ and $y > 0$.

Proof. From (2) and (5) we obtain

$$(\mathrm{Im}\ z)\left|-\mathrm{i}\frac{\partial}{\partial y}[\ln f(x + \mathrm{i}y)]\right| \le 2$$

or equivalently

$$\left|\frac{\partial}{\partial y}[\ln |f(x + \mathrm{i}y)|] + \mathrm{i}\frac{\partial}{\partial y}[\arg f(x + \mathrm{i}y)]\right| \le \frac{2}{y}.$$

This last inequality implies

$$\left|\frac{\partial}{\partial y}[\ln|f(x+\mathrm{i}y)|]\right| \leq \frac{2}{y}, \text{ for all } x \in \mathbb{R} \text{ and } y > 0.$$

If $y \in (0,1)$ we have

$$-\int_y^1 \frac{2}{y}\mathrm{d}y \leq \int_y^1 \frac{\partial}{\partial y}[\ln|f(x+\mathrm{i}y)|]\mathrm{d}y \leq \int_y^1 \frac{2}{y}\mathrm{d}y.$$

Thus we obtain

$$\text{(9)} \qquad \frac{1}{y^2} \leq \left|\frac{f(x+\mathrm{i}y)}{f(x+\mathrm{i})}\right| \leq y^2, \text{ for all } x \in \mathbb{R} \text{ and } y \in (0,1).$$

If $y \in (1,\infty)$ then

$$-\int_1^y \frac{2}{y}\mathrm{d}y \leq \int_1^y \frac{\partial}{\partial y}[\ln|f(x+\mathrm{i}y)|]\mathrm{d}y \leq \int_1^y \frac{2}{y}\mathrm{d}y$$

and therefore

$$\text{(10)} \qquad \frac{1}{y^2} \leq \left|\frac{f(x+\mathrm{i}y)}{f(x+\mathrm{i})}\right| \leq y^2, \text{ for all } x \in \mathbb{R} \text{ and } y \geq 1.$$

Inequalities (7) follow from (9) and (10).

In the same way, from (2) and (6) we obtain

$$\left|\frac{\partial}{\partial x}[\ln|f(x+\mathrm{i}y)|]\right| \leq \frac{2}{y}$$

and then

$$\mathrm{e}^{-2\frac{|x|}{y}} \leq \left|\frac{f(x+\mathrm{i}y)}{f(iy)}\right| \leq \mathrm{e}^{2\frac{|x|}{y}}, \text{ for all } x \in \mathbb{R} \text{ and } y > 0.$$

If $y = 1$ in these last inequalities, then we have (8). □

Remark 1. In Theorem 3, inequalities (7) and (8) are sharp. The extremal function is $f(z) = z^2$, $z \in D$.

From inequalities (7) and (8) we obtain the next corollary.

Corollary 1. *If $f \in H(D)$ is an univalent function in D such that $f(z) \neq 0$, for all $z \in D$, then*

$$m(y)\mathrm{e}^{-2|x|} \leq \left|\frac{f(z)}{f(\mathrm{i})}\right| \leq M(y)\mathrm{e}^{2|x|},$$

for all $z = x + \mathrm{i}y \in D$, where

$$m(y) = \min\left\{y^2, \frac{1}{y^2}\right\} \quad \text{and} \quad M(y) = \max\left\{y^2, \frac{1}{y^2}\right\}.$$

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