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THEORY OF SUPERORDINATIONS FOR SEVERAL COMPLEX VARIABLES

VERONICA NECHITA

Abstract. Let D be any set of $\mathbb{C}^n$, let p be holomorphic in the unit ball B^n and let $\varphi : \mathbb{C}^n \times \mathbb{C}^n \times B^n \to \mathbb{C}^n$. In this article we consider the problem of determining properties of functions p that satisfy the superordination

$$D \subset \left\{ \varphi \left(p(\zeta), [(Dp(\zeta))^*]^{-1}(\zeta); \zeta \right) : \zeta \in B^n \right\}.$$

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1. INTRODUCTION

Let Ω be any set in the complex plane, let p be analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$. The theory of differential subordinations in the complex plane is by now a classical problem in geometric function theory and deals with the problem of finding properties for functions p that satisfy the subordination

$$\left\{ \psi \left(p(z), zp'(z), z^2 p''(z); z \right) : z \in U \right\} \subset \Omega.$$

Recently, S.S. Miller and P.T. Mocanu [6] considered the dual problem, that of determining properties for the functions p that satisfy the superordination

$$\Omega \subset \left\{ \psi \left(p(z), zp'(z), z^2 p''(z); z \right) : z \in U \right\}.$$

Let D be any set of $\mathbb{C}^n$, let p be holomorphic in the unit ball B^n and let $\varphi : \mathbb{C}^n \times \mathbb{C}^n \times B^n \to \mathbb{C}^n$. In the last years there was a constant effort to extend the results from the complex plane to several complex variables. One of the generalizations is due to P. Curt [1] and deals with differential subordinations of the form

$$\left\{ \varphi \left(p(\zeta), [(Dp(\zeta))^*]^{-1}(\zeta); \zeta \right) : \zeta \in B^n \right\} \subset D.$$

In this article we consider the problem of determining properties of functions p that satisfy the superordination

$$D \subset \left\{ \varphi \left(p(\zeta), [(Dp(\zeta))^*]^{-1}(\zeta); \zeta \right) : \zeta \in B^n \right\}.$$

2. PRELIMINARIES

We denote by $\mathbb{C}^n$ the Euclidean space of n complex variables with the standard inner product

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w_j},\ z, w \in \mathbb{C}^n,$$

and the norm $\|z\| = \langle z, z\rangle^{1/2}$, $z \in \mathbb{C}^n$. Vectors and matrices marked with the symbols $'$ and * denote the transposed and transposed conjugate vector and matrix respectively.

The open set $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r^n, while the unit ball is abbreviated by $B_1^n = B^n$. The class of holomorphic mappings $f : B^n \to \mathbb{C}^n$ is denoted by $\mathcal{H}(B^n)$.

A mapping $f \in \mathcal{H}(B^n)$ is called locally biholomorphic on B^n if its Fréchét derivative $Df(z)$, as an element of $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$, is nonsingular at each point $z \in B^n$. A mapping $f \in \mathcal{H}(B^n)$ is called biholomorphic if the inverse mapping is holomorphic on $f(B^n)$. If $D^2 f(z)$ represents the Fréchét derivative of the second order of $f \in \mathcal{H}(B^n)$ at the point z, then $D^2 f(z)$ is a continuous bilinear operator from $\mathbb{C}^n \times \mathbb{C}^n$ into $\mathbb{C}^n$, while its restriction $D^2 f(z)(u, \cdot)$ to $u \times \mathbb{C}^n$ belongs to $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$.

Let f and g be members of $\mathcal{H}(B^n)$. The mapping f is said to be subordinate to g, or the mapping g is said to be superordinate to f, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a mapping $w \in \mathcal{H}(B^n)$, with $w(0) = 0$ and $\|w(z)\| < 1$, and such that $f(z) = g(w(z))$.

From this definition we see that if $f \prec g$, then $f(0) = g(0)$ and $f(B^n) \subseteq g(B^n)$. If in addition g is biholomorphic, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(B^n) \subseteq g(B^n)$.

By using an extended version of the Schwarz Lemma it is easy to prove that if $f \prec g$, then $f(B_r^n) \subset g(B_r^n)$, for all $0 < r < 1$.

In this paper we will use a reformulation of [1, Lemma 2] from the theory of differential subordinations of several complex variables.

LEMMA 1. *Let $p \in \mathcal{H}\left(\overline{B^n}\right)$ be a biholomorphic mapping, $q \in \mathcal{H}(B^n)$ locally biholomorphic on B^n, with $q(0) = p(0)$. If p is not superordinated to q, then there exist $t > 1$ and points $z_0 \in B^n$, $\zeta_0 \in \overline{B^n}$, with $\|\zeta_0\| = 1$ for which*

(i) $q(z_0) = p(\zeta_0)$;

(ii) $t\left[(Dq(z_0))^*\right]^{-1}(z_0) = \left[(Dp(\zeta_0))^*\right]^{-1}(\zeta_0)$;

(iii) *the inequality*

$$t[\|u\|^2 - \mathrm{Re}\langle [Dq(z_0)]^{-1} D^2 q(z_0)(u, u), z_0\rangle] \geq \|w\|^2 - \mathrm{Re}\langle [Dp(\zeta_0)]^{-1} D^2 p(\zeta_0)(w, w), \zeta_0\rangle,$$

holds for all $u \in \mathbb{C}^n \setminus \{0\}$ with $\mathrm{Re}\langle u, z_0\rangle = 0$, where $w = [Dp(\zeta_0)]^{-1} Dp(\zeta_0)\, u$.

3. ADMISSIBLE FUNCTIONS AND A FUNDAMENTAL RESULT

We next define the class of admissible mappings.

DEFINITION 1. *Let D be a set in $\mathbb{C}^n$ and $q \in \mathcal{H}(B^n)$ a locally biholomorphic mapping on B^n. The class of admissible mappings $\Phi[D, q]$ consists of those functions $\varphi : \mathbb{C}^n \times \mathbb{C}^n \times B^n \to \mathbb{C}^n$ that satisfy*

$$\varphi(x, y; \zeta) \in D,$$

whenever $x = q(z)$, $y = t\left[(Dq(z))^*\right]^{-1}(z)$, $z \in B^n$, $\zeta \in \overline{B^n}$, $\|\zeta\| = 1$ and $t > 1$.

The next theorem is a foundation result in the theory of differential superordinations for functions of several variables. The proof is very short because of the use of Lemma 1 and the very special conditions in the definition of the class of admissible functions $\Phi[D, q]$.

THEOREM 1. *Let D be a set in $\mathbb{C}^n$, $q \in \mathcal{H}(B^n)$ a locally biholomorphic mapping on B^n and $\varphi \in \Phi[D, q]$. If $p \in \mathcal{H}\left(\overline{B^n}\right)$ is a biholomorphic mapping on $\overline{B^n}$ such that $p(0) = q(0)$ and $\varphi\left(p(\zeta), \left[(Dp(\zeta))^*\right]^{-1}(\zeta); \zeta\right)$ is injective on $\overline{B^n}$, then*

$$D \subset \left\{\varphi\left(p(\zeta), \left[(Dp(\zeta))^*\right]^{-1}(\zeta); \zeta\right) : \zeta \in B^n\right\} \qquad (1)$$

implies $q \prec p$.

Proof. Assume $q \not\prec p$. By Lemma 1, there exist two points $z_0 \in B^n$, $\zeta_0 \in \overline{B^n}$, with $\|\zeta_0\| = 1$ and an $t > 1$ that satisfy the conditions (i)-(iii) of Lemma 1. Using these conditions with $x = p(\zeta_0)$, $y = \left[(Dp(\zeta_0))^*\right]^{-1}(\zeta_0)$ and $\zeta = \zeta_0$ in Definition 1 we obtain

$$\varphi(p(\zeta_0), \left[(Dp(\zeta_0))^*\right]^{-1}(\zeta_0), \zeta_0) \in D.$$

Since this contradicts (1) we must have $q \prec p$. □

4. EXAMPLES

If we choose $q(z) = Mz$ $(M > 0)$ for all $z \in B^n$ in Definition 1 and Theorem 1, we obtain:

COROLLARY 1. *Let D be a set in $\mathbb{C}^n$ and $\varphi : \mathbb{C}^n \times \mathbb{C}^n \times B^n \to \mathbb{C}^n$ such that $\varphi\left(Mz, \frac{t}{M}z; \zeta\right) \in D$, for $z \in B^n$, $\zeta \in \overline{B^n}$, $\|\zeta\| = 1$ and $t > 1$.*

If $p \in \mathcal{H}\left(\overline{B^n}\right)$ is a biholomorphic mapping on $\overline{B^n}$ such that $p(0) = 0$ and $\varphi\left(p(\zeta), \left[(Dp(\zeta))^*\right]^{-1}(\zeta); \zeta\right)$ is injective on $\overline{B^n}$, then

$$D \subset \left\{\varphi\left(p(\zeta), \left[(Dp(\zeta))^*\right]^{-1}(\zeta); \zeta\right) : \zeta \in B^n\right\} \qquad (2)$$

implies $B^n_M \subseteq p(B^n)$.

COROLLARY 2. *Let M be a real and positive number and $\lambda \geq 0$. If $p \in \mathcal{H}\left(\overline{B^n}\right)$ is a biholomorphic mapping on $\overline{B^n}$ such that $p(0) = 0$ and $p(\zeta) - \lambda\left[(Dp(\zeta))^*\right]^{-1}(\zeta)$ is injective on $\overline{B^n}$, then*

$$B_M^n \subset \left\{p(\zeta) - \lambda\left[(Dp(\zeta))^*\right]^{-1}(\zeta) : \zeta \in B^n\right\}$$

implies $B_M^n \subseteq p(B^n)$.

EXAMPLE 1. *Let M be a positive real number, $\lambda \in (0,1)$, let B^2 be the unit ball of $\mathbb{C}^2$ and $p \in \mathcal{H}\left(\overline{B^2}\right)$ the biholomorphic mapping given by*

$$p(\zeta) = \zeta.$$

The function $p(\zeta) - \lambda\left[(Dp(\zeta))^*\right]^{-1}(\zeta) = (1-\lambda)\zeta$ is injective on $\overline{B^2}$.

EXAMPLE 2. *Let M be a real and positive number, $\lambda > 0$ and let p_1, p_2 be complex univalent functions defined in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, $R > 1$, such that*

$$\left[\left|p_i'(z)\right|^2 - \lambda\right] \mathrm{Re} p_i'(z) > \lambda |z| \left|p_i''(z)\right|, \text{ for } i = 1,2 \text{ and all } z \in U_R.$$

We define $p : B^2 \to \mathbb{C}^2$, $p(\zeta) = (p_1(\zeta_1), p_2(\zeta_2))'$. Since p satisfies the conditions of the Corollary 2, from the inclusion

$$B_M^2 \subset \left\{p(\zeta) - \lambda\left[(Dp(\zeta))^*\right]^{-1}(\zeta) : \zeta \in B^2\right\},$$

it implies $B_M^2 \subseteq p\left(B^2\right)$.

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Faculty of Mathematics and Comp. Science
"Babeş-Bolyai" University
Str. M. Kogălniceanu 1
RO-400084 Cluj-Napoca, România
E-mail: vero@math.ubbcluj.ro