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EXTENSION OF LINEAR OPERATORS, DISTANCED CONVEX SETS AND THE MOMENT PROBLEM

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Abstract. One applies an extension theorem of linear operators ([10, Theorem 5, p. 969]) to the classical moment problem in spaces of continuous functions on a compact interval and in spaces of analytic functions. One proves that certain conditions (which are often fulfilled) are sufficient for the existence of some solutions of some moment problems. Our solutions satisfies some sandwich type conditions. One of these conditions (the inequalities (5)) and the fact that equalities (4) hold only for $j \geq 1$ are in a way unusual with respect to some other moment problems. We exploit the notions of distanced convex sets and positive sequence on an interval. One solves an operator-valued moment problem.

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1. INTRODUCTION

We recall some general facts concerning the classical moment problem. Let X be a space of real or complex functions defined on a compact subset in $\mathbf{R}^n$ which contains the polynomials $x_j(t) = t^j$, $j \in \mathbf{N}^n$ and let $\{y_j\}_{j\in\mathbf{N}^n}$ a sequence of real (respectively complex) numbers. The problem is: find necessary and sufficient (or only sufficient) conditions on $\{y_j\}_{j\in\mathbf{N}^n}$ for the existence of a linear functional f on X such that the moment conditions

$$f(x_j) = y_j, \quad j \in \mathbf{N}^n$$

are satisfied and such that f have some other sandwich type properties, which generalize the continuity and positivity of the linear functional f. If we define $f_0 : Sp\,\{x_j; j \in \mathbf{N}^*\} \to \mathbf{R}$ (or $\mathbf{C}$) by

$$f_0\left(\sum_{j\in F\subset\mathbf{N}^n} \lambda_j x_j\right) := \sum_{j\in F} \lambda_j y_j$$

(finite sums), then it is clear that to solve the moment problem means to find conditions upon $\{y_j\}_{j\in\mathbf{N}^n}$ such that f_0 may be extended to a linear functional f defined on the whole space X, such that some sandwich conditions are satisfied. The y_j, $j \in \mathbf{N}^n$ are called moments since they generalize the classical moments (see [2]).

The general extension result stated in section 2 from below enables us to find sufficient conditions for the existence of solutions of some moment problems which may lead (via measure theory) to some solutions of some Markov moment type problems (see [5]).

2. A GENERAL EXTENSION THEOREM FOR LINEAR OPERATORS

THEOREM 2.1. [10, Theorem 5, p. 969] *Let X be a locally convex space, Y an order-complete vector lattice with strong order unit u_0 and let X_0 be a vector subspace of X. Let $A \subset X$ be a convex subset such that the following two conditions are fulfilled:*

(a) *There exists a neighbourhood V of the origin such that $(X_0+V)\cap A = \Phi$ (A and X_0 are distanced).*

(b) *A is bounded.*

Then for any equicontinuous family of linear operators $\{f_i\}_{i\in I} \subset L(X_0, Y)$ and for any $\tilde{y} \in Y_+ \setminus \{0\}$, there exists an equicontinuous family $\{\tilde{f}_i\}_{i\in I} \subset L(X,Y)$ such that $\tilde{f}_i|_{X_0} = f_i$ and $\tilde{f}_i|_A \geq \tilde{y}$, $i \in I$.

Moreover, let u_0 be a strong unit in Y and V a convex, circled neighbourhood of the origin, with the properties

$$f_i(V \cap X_0) \subset [-u_0, u_0], \tag{1}$$

$$(X_0 + V) \cap A = \Phi. \tag{2}$$

If we denote by p_V the Minkowski functional attached to V and we choose $0 < \alpha \in \mathbf{R}$ such that $p_V|_A \leq \alpha$ and $\alpha_1 > 0$ such that $\tilde{y} \leq \alpha_1 u_0$, then the following relations hold

$$\tilde{f}_i(x) \leq (1 + \alpha + \alpha_1)p_V(x)u_0,\ x \in X,\ i \in I. \tag{3}$$

3. MAIN RESULTS

THEOREM 3.1. *Let $0 < b \in \mathbf{R}$, $X := C([0,b])$, $x_j(t) = t^j$, $j \in \mathbf{N}$, $j \geq 1$, $t \in [0,b]$, $\{\varphi_k : k \in \mathbf{N}\} \subset X$, $||\varphi_k|| \leq 1$, $\varphi_0 \equiv 1$, $\varphi_k(0) = 1$, $k \in \mathbf{N}$. Let Y be an order complete vector lattice with strong unit u_0, and let $\{y_1, y_2, \ldots\} \subset Y$ be such that the sequence $\{u_0, y_1, y_2, \ldots\}$ is positive on $[0,b]$ ($\sum_{j=0}^{n} \lambda_j t^j \geq 0$ $\forall t \in [0,b] \Rightarrow \lambda_0 u_0 + \sum_{j=1}^{n} \lambda_j y_j \geq 0$ in Y, $n \in \mathbf{N}$, $\lambda_j \in \mathbf{R}$).*

Then, for any $\alpha_1 \in \mathbf{R}_+$, there exists $f \in L(X,Y)$ such that

$$f(x_j) = y_j, \quad j \in \mathbf{N}, \quad j \geq 1, \tag{4}$$

$$f(\varphi_k) \geq \alpha_1 u_0, \quad k \in \mathbf{N}, \tag{5}$$

$$f(x) \leq (2 + \alpha_1)||x||u_0,\ x \in X. \tag{6}$$

Moreover, if $\alpha_1 \geq 1$ and if Y is endowed with a linear topology such that the positive cone Y_+ is closed and normal, then f is continuous and positive.

Proof. We apply Theorem 2.1 to $X = C([0,b])$, $X_0 = Sp\{x_j; j \in \mathbf{N}, j \geq 1\}$, $A := co\{\varphi_k : k \in \mathbf{N}\}$. For any $p_0 \in X_0$ and any $a \in A$, we have

$$||p_0 - a|| \geq |p_0(0) - a(0)| = 1,$$

whence $d(X_0, A) \geq 1$. This implies $(X_0 + B(0,1)) \cap A = \Phi$, where $B(0,1) := \{x \in X; ||x|| < 1\}$. We take $V := B(0,1)$ in Theorem 2.1. Thus $p_V = ||\ ||$. On the other hand, $||\varphi_k|| \leq 1$, $k \in \mathbf{N} \Rightarrow p_V|_A = ||\ |||_A \leq 1$. So, we can take $\alpha := 1$ in Theorem 2.1. We also take $\tilde{y} := \alpha_1 u_0$, $f_0\left(\sum_{j=1}^n \lambda_j x_j\right) := \sum_{j=1}^n \lambda_j y_j \quad (I = \{0\})$. Now we check (1). Let $\sum_{j=1}^n \lambda_j x_j \in X_0 \cap V = X_0 \cap B(0,1)$. Then we have

$$\sup\left\{\left|\sum_{j=1}^n \lambda_j t^j\right|;\ t \in [0,b]\right\} < 1,$$

i.e. $\sum_{j=1}^n \lambda_j t^j + 1 > 0, \quad t \in [0,b]$ and $1 - \sum_{j=1}^n \lambda_j t^j > 0$, $t \in [0,b]$. Since the sequence $\{u_0, y_1, y_2, \ldots\}$ is supposed to be positive on $[0,b]$, these relations lead to

$$u_0 + \sum_{j=1}^n \lambda_j y_j \geq 0 \quad \text{and} \quad u_0 - \sum_{j=1}^n \lambda_j y_j \geq 0$$

which mean

$$f_0\left(\sum_{j=1}^n \lambda_j x_j\right) = \sum_{j=1}^n \lambda_j y_j \in [-u_0, u_0],$$

i.e. (1).

By Theorem 2.1, there exists $\tilde{f_0} =: f \in L(X,Y)$ such that $f|_{X_0} = f_0$, $f|_A \geq \alpha_1 u_0$, $f(x) \leq (1 + 1 + \alpha_1)||x||u_0$, $x \in X$. These relations imply (4), (5), (6). Let now $\alpha_1 \geq 1$, and let Y be endowed with a topology such that Y_+ is closed and normal. We have to prove that f is continuous and positive. From (6) (written also for $-x$ instead of x) and from the fact that Y_+ is normal, we deduce the continuity of f. To prove that f is also positive, it is sufficient to show that $f(p) \geq 0$ for any positive polynomial p (then one uses Weierstrass-Bernstein theorem and the fact that Y_+ is closed). So, let $p(t) = \lambda_0 + \lambda_1 t + \ldots + \lambda_n t^n \geq 0$, $\forall t \in [0,b]$. Since $\{u_0, y_1, y_2, \ldots\}$ is positive on $[0,b]$, we deduce $\lambda_0 u_0 + \lambda_1 y_1 + \ldots + \lambda_n y_n \geq 0$, i.e. $\sum_{j=1}^n \lambda_j y_j \geq -\lambda_0 u_0$ in Y. On the other hand, since we have supposed that $\varphi_0 \equiv 1$ and $\alpha_1 \geq 1$, we get

$$f(\lambda_0 \varphi_0 + \lambda_1 x_1 + \ldots + \lambda_n x_n) = \lambda_0 f(\varphi_0) + \sum_{j=1}^n \lambda_j y_j \geq$$

$$\geq \lambda_0 f(\varphi_0) - \lambda_0 u_0 \overset{(5)}{\geq} \lambda_0(\alpha_1 u_0 - u_0) = \lambda_0(\alpha_1 - 1)u_0 \geq 0$$

i.e. $f(p) \geq 0$. The proof is complete. □

COROLLARY 3.2. *Let H be a Hilbert space and let $A \in \mathcal{A}(H)$ = the set of all selfadjoint operators applying H into H. We suppose that A is positive ($< A(h), h > \geq 0 \; \forall h \in H$). We denote*

$$\mathcal{A}_1 := \{U \in \mathcal{A}(H);\; UA = AU\}, \qquad Y := \{U \in \mathcal{A}_1;\; UV = VU,\; \forall V \in \mathcal{A}_1\}.$$

Let $b \in \mathbf{R}_+$ such that $S(A) \subset [0, b]$, where $S(A)$ is the spectrum of A. Let $\alpha_1 \geq 1$. Then there exists an increasing function $\sigma : [0, b] \to Y$ such that

$$\int_0^b t^j \mathrm{d}\sigma(t) = A^j, \quad j \in \mathbf{N}, \quad j \geq 1 \tag{4'}$$

$$\int_0^b \mathrm{e}^{-kt} \mathrm{d}\sigma(t) \geq \alpha_1 I, \quad k \in \mathbf{N} \tag{5'}$$

$$\int_0^b x(t)\mathrm{d}\sigma(t) \leq (2 + \alpha_1)||x||I,\; x \in C([0, b]), \tag{6'}$$

where I is the identity operator.

In particular, for such a function σ we have

$$\alpha_1 + 1 \leq ||\sigma(b) - \sigma(0)|| + ||\exp(-kA)||,\; k \in \mathbf{N} \tag{7}$$

Proof. We apply Theorem 3.1 to $\varphi_k(t) = \mathrm{e}^{-kt}$, $k \in \mathbf{N}$, $t \in [0, b]$, Y defined above, $u_0 = I$ (it is well-known that Y is a complete vector lattice with strong unit $u_0 = I$ (see [4]). We have to show that the sequence $\{I, A, A^2, \ldots\}$ is positive on $[0, b] \supset S(A)$. This is a consequence of the existence of the spectral measure attached to A, which is positive since A is selfadjoint. By Theorem 3.1, there exists $f \in L(X, Y)$ such that (4), (5) and (6) hold. Since $\alpha_1 \geq 1$ and the positive cone Y_+ is closed and normal, f is continuous and positive. Using the representation theorem of linear and (τo) bounded operators $f : C([0, b]) \to Y$ (see [4], p. 272), we deduce the existence of a function of bounded variation $\sigma : [0, b] \to Y$ with the properties (4′), (5′), (6′). Since f is positive, σ is increasing. Now we prove (7). We have

$$\begin{aligned}
\alpha_1 I \;&\overset{(5')}{\leq}\; \int_0^b \mathrm{e}^{-kt}\mathrm{d}\sigma(t) = \sum_{m=0}^{\infty} \frac{(-k)^m}{m!} \int_0^b t^m \mathrm{d}\sigma(t) \overset{(4')}{=} \\
&= \;\sigma(b) - \sigma(0) + \sum_{m=1}^{\infty} \frac{(-k)^m}{m!} A^m = \\
&= \;\sigma(b) - \sigma(0) - I + \sum_{m=0}^{\infty} \frac{(-k)^m}{m!} A^m = \\
&= \;\sigma(b) - \sigma(0) - I + \exp(-kA),\; k \in \mathbf{N}
\end{aligned}$$

This is equivalent to

$$< (\alpha_1 + 1)I(h), h > \leq < (\sigma(b) - \sigma(0)(h), h > + \\ + < \exp(-kA)(h), h >, \; h \in N, \; k \in \mathbf{N},$$

which implies

$$(\alpha_1 + 1)||h||^2 \leq ||\sigma(b) - \sigma(0)|| \; ||h||^2 + ||\exp(-kA)|| \; ||h||^2,$$

$h \in H$, $k \in \mathbf{N}$, which is equivalent to (7).

The proof is completed. □

Next we consider the case when the moments y_j, $j \in \mathbf{N}$ are real numbers ($Y = \mathbf{R}$).

Via Jensen inequality, we obtain some inequalities connecting the terms of a sequence, which is positive on $[0, b]$, to the second term of the sequence (see inequalities (8) from below).

Corollary 3.3. *Let* $\{1, y_1, y_2, \ldots\} \subset \mathbf{R}$ *be a positive sequence on* $[0, b]$*. Then the following inequalities hold*

$$y_1 \leq 3^{(j-1)/j} \cdot y_j^{1/j}, \; j \in \mathbf{N}, \; j \geq 1 \tag{8}$$

(in particular, $y_1 \leq 3 \liminf y_j^{1/j}$*).*

Proof. We apply Theorem 3.1 to $Y = \mathbf{R}$, $u_0 = 1$, $\varphi_k(t) = \mathrm{e}^{-kt}$, $t \in [0, b]$, $\alpha_1 \geq 1$. The sequence $\{1, y_1, y_2, \ldots\}$ being supposed to be positive on $[0, b]$, by Theorem 3.1, there exists a linear functional $f \in (C([0, b]))^*$ such that (4), (5) and (6) hold. Using the representation theorem of linear positive functionals $f : C([0, b]) \to \mathbf{R}$ by measures $\mathrm{d}\sigma$ with $\sigma : [0, b] \to \mathbf{R}$ increasing function, there exists such a function σ such that

$$\int_0^b t^j \mathrm{d}\sigma(t) = y_j, \quad j \in \mathbf{N}, \quad j \geq 1, \tag{4''}$$

$$\int_0^b \exp(-kt) \mathrm{d}\sigma(t) \geq \alpha_1, \quad k \in \mathbf{N}, \tag{5''}$$

$$\int_0^b x(t) \mathrm{d}\sigma(t) \leq (2 + \alpha_1)||x||, \; x \in C([0, b]) \tag{6''}$$

hold. By (5″), σ is not constant. Now we apply the following particular variant of the Jensen's inequality

$$\int_a^b h(t)p(t)\mathrm{d}\sigma(t) \geq \left(\int_a^b p(t)\mathrm{d}\sigma(t)\right) \cdot h\left(\frac{\int_a^b tp(t)\mathrm{d}\sigma(t)}{\int_a^b p(t)\mathrm{d}\sigma(t)}\right),$$

(where $\sigma : [a, b] \to \mathbf{R}$ is an increasing nonconstant function, h is continuous and convex, p is continuous and nonnegatively, $p \not\equiv 0$) to $h(t) = t^j$, $j \geq 1$, $p(t) = 1$ $\forall t \in [0, b]$. We find

$$\begin{aligned} y_j &\overset{(4'')}{=} \int_0^b t^j \mathrm{d}\sigma(t) \geq \int_0^b \mathrm{d}\sigma(t) \left[\frac{\int_0^b t \mathrm{d}\sigma(t)}{\int_0^b \mathrm{d}\sigma(t)} \right]^j \\ &\overset{(4'')}{=} [\sigma(b) - \sigma(0)] \cdot \left[\frac{y_1}{\sigma(b) - \sigma(0)} \right]^j = \frac{y_1^j}{[\sigma(b) - \sigma(0)]^{j-1}}, \end{aligned}$$

i.e. $y_j [\sigma(b) - \sigma(0)]^{j-1} \geq y_1^j$, which leads to

$$y_1 \leq [\sigma(b) - \sigma(0)]^{(j-1)/j} y_j^{1/j} = \left(\int_0^b \mathrm{d}\sigma(t) \right)^{(j-1)/j} y_j^{1/j} \overset{(6'')}{\leq} (2 + \alpha_1)^{(j-1)/j} y_j^{1/j},$$

the last inequality being valid for any $\alpha_1 \geq 1$. Writing it for $\alpha_1 = 1$, we find (8). □

REMARK 3.4. Using Taylor formula and finite sums $\sum_{n=1}^{p} \frac{(-1)^n k^n}{n!} t^n$ with $p \in \mathbf{N}$ odd number, it is easy to prove that if $\{1, y_1, y_2, \ldots\}$ is a positive sequence on $[0, b]$, then we have $\sum_{n=1}^{p} \frac{(-1)^n k^n}{n!} y_n \leq 0$. Similarly, for p even number, one obtains $\sum_{n=1}^{p} \frac{(-1)^n k^n}{n!} y_n \geq -1$. From these two inequalities, making $p \to \infty$, we get

$$-1 \leq \sum_{n=1}^{\infty} \frac{(-1)^n k^n}{n!} y_n \leq 0.$$

Note that for these inequalities it is not necessary to use the fact that $\{y_1, y_2, \ldots\}$ is a moment sequence (by Theorem 3.1).

Next we consider a moment problem in a space X of analytic functions on an open disk, which are continuous on the closed disk. Let $b > 0$ and $X := A_b$ the space of all functions x which may be represented as an absolutely convergent series $x(z) = \sum_{j=0}^{\infty} \lambda_j z^j$, $|z| < b$, $\lambda_j \in \mathbf{R}$, x being continuous on the closed disk $|z| \leq b$. For $x \in X$, we note $||x|| = \sup\{|x(z)|;\ |z| \leq b\}$. Let $x_j \in X$, $x_j(z) = z^j$, $j \in \mathbf{N}$. Let $Y = L^{\infty}(\Omega)$, where (Ω, μ) is a measurable space, the measure μ being positive. We denote by $u_0 \in Y$ the function $u_0(\omega) = 1$, $\omega \in \Omega$ (u_0 is a strong order unit in Y endowed with the usual cone Y_+). For $y \in Y$, we note $||y||_{\infty} = \mathrm{esssup} y$. With these notations, from theorem 2.1 we deduce

THEOREM 3.5. *Let $b > 1$, $\{\varphi_k : k \in \mathbf{N}\} \subset X$ such that $||\varphi_k|| \leq M$, $\varphi_k(0) = 1$, $k \in \mathbf{N}$. Let $\{y_j : j \in \mathbf{N},\ j \geq 1\} \subset Y$ be a sequence such that $||y_j||_\infty \leq b - 1$, $j \geq 1$.*

Then for any $\tilde{y} \in Y_+$, there exists $f \in L(X, Y)$ such that

$$f(x_j) = y_j, \quad j \in \mathbf{N}, \quad j \geq 1 \tag{4'''}$$

$$f(\varphi_k) \geq \tilde{y}, \quad k \in \mathbf{N} \tag{5'''}$$

$$f(x) \leq (1 + M + ||\tilde{y}||_\infty)||x||u_0,\ x \in X \tag{6'''}$$

Proof. We apply Theorem 2.1 to $X_0 := Sp\{x_j, j \geq 1\}$, $A := co\{\varphi_k; k \in \mathbf{N}\}$, $f_0\left(\sum_{j=1}^{n} \lambda_j x_j\right) := \sum_{j=1}^{n} \lambda_j y_j$. By the conditions upon φ_k, $k \in \mathbf{N}$, it follows that

$$d(X_0, A) = \inf\{||x - a||;\ x \in X_0,\ a \in A\} \geq 1$$

Thus we get $(X_0 + B(0,1)) \cap A = \Phi$, so we can apply Theorem 2.1. to $V = B(0,1)$. This implies $p_V = ||\ ||$. We check (1)

$$x \in B(0,1) \cap X_0 \Rightarrow x = \sum_{j=1}^{n} \lambda_j x_j,\ ||x|| < 1.$$

Using the Cauchy inequalities for the analytic function x, we find

$$|\lambda_j| \leq ||x||/b^j < 1/b^j,\ j \in \{1, 2, \ldots, n\}.$$

On the other hand, for any $y \in Y = L^\infty(\Omega)$, we have $|y(\omega)| \leq ||y||_\infty \cdot u_0(\omega)$ a.e. in Ω, whence $|y| \leq ||y||_\infty u_0$. These relations and the hypothesis $||y_j||_\infty \leq b - 1,\ j \geq 1$ lead to

$$\begin{aligned} \left|\sum_{j=1}^{n} \lambda_j y_j\right| &\leq \sum_{j=1}^{n} |\lambda_j|\,|y_j| \leq \left(\sum_{j=1}^{n} (1/b^j)||y_j||_\infty\right) u_0 \leq \\ &\leq \left(\sum_{j=1}^{\infty} 1/b^j\right)(b-1)u_0 = u_0, \end{aligned}$$

i.e. $\sum_{j=1}^{n} \lambda_j x_j \in B(0,1) \cap X_0 \Rightarrow f_0\left(\sum_{j=1}^{n} \lambda_j x_j\right) = \sum_{j=1}^{n} \lambda_j y_j \in [-u_0, u_0]$.

On the other hand, we remark that $p_V|_A = ||\ |||_A \leq M$, so we may take in Theorem 2.1 $\alpha := M$. One also remarks that $\tilde{y} \leq ||\tilde{y}||_\infty u_0$, hence we may take $\alpha_1 := ||\tilde{y}||_\infty$. The conclusion follows. □

A similar result may be proved for spaces X of analytic functions on a n-dimensional open polydisk D, continuous on the closed polydisk $\bar{D}$.

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