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ON THE MODULAR INTEGRALS

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Abstract. Let f be an entire modular integral on $\Gamma(1)$ of weight k. We investigate necessary and sufficient conditions for $f(\tau^m)$ to be a modular integral on $\Gamma(1)$ of weight mk. We deduce some relations among the Mellin transforms of functions $f(\tau)$, $f(\tau^m)$ and $f(\tau^m/m', \chi)$. We rewrite without proof some theorems from [4] and [5] for the function $f(\tau^m)$ and the subgroup $\Gamma_*^0(N)$.

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1. INTRODUCTION

We shall use H to denote the upper half-plane, and $\Gamma(1)$ for the modular group; τ is a complex variable in H. Let $N \geq 1$ be an integer and put

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : c \equiv 0 \pmod N \right\}.$$

Let $U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and Let $\Gamma^0(N)$ be the subgroup defined by $b \equiv 0 \pmod N$. In fact $V\Gamma_0(N)V^{-1} = \Gamma^0(N)$. Let $\omega(N)$ be the inversion defined by $\omega(N) : \tau \mapsto -1/N\tau$; as a matrix, $\omega(N) = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$. Note that $\omega(N) \notin \Gamma(1)$ if $N > 1$. Let $\Gamma_0^*(N) = \langle \Gamma_0(N); \omega(N) \rangle$, that is, $\Gamma_0^*(N)$ is the larger group obtained by extending $\Gamma_0(N)$ by the inversion $\omega(N)$. We define $\Gamma_*^0(N) = \langle \Gamma^0(N); \omega(N) \rangle$ (see [1] and [5]).

2. MELLIN TRANSFORMS OF MODULAR INTEGRALS ON $\Gamma(1)$

Let f be an entire modular integral (entire MI for short) on $\Gamma(1)$ of weight k, with multiplier system (MS for short) υ. That is to say:

(i) f satisfies the conditions

$$f(\tau + 1) = f(\tau), \qquad \tau^{-k} f(-\frac{1}{\tau}) = \upsilon(V) f(\tau) + q(\tau); \tag{1}$$

(ii) f is holomorphic in H;

(iii) f has a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{2\pi i n \tau}, \tag{2}$$

where $k \in 2\mathbb{Z}$ and $q(\tau)$ is a rational period function (RPF).

It follows from these three conditions that $a_n = O(n^\gamma)$, $n \to +\infty$ for some $\gamma > 0$, and this in turn guarantees the absolute convergence of the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ in the half-plane $\operatorname{Re} s > 1 + \gamma$. This series arises naturally from term-by-term integration when one forms the Mellin transform

$$\Phi_f(s) = \int_0^\infty (f(\mathrm{i}y) - a_0) y^s \frac{\mathrm{d}y}{y} = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} a_n n^{-s}, \tag{3}$$

of $f(\tau) - a_0$. Note that $\Phi_f(s)$, like the Dirichlet series, is holomorphic in $\operatorname{Re} s > 1 + \gamma$. The classic work of Hecke [1], [2] shows that if f is an entire modular form (that is, if $q = 0$ in (1)), then $\Phi_f(s)$ has certain desirable properties, the most striking among them being the functional equation

$$\Phi_f(k - s) = (-1)^{k/2}\Phi_f(s).$$

The Mellin transform of an entire MI in $\Gamma(1)$ with RPF q has precisely the same functional equation as does the Mellin transform of an entire modular form on $\Gamma(1)$.

In addition to $\Phi_f(s)$ we consider the "twisted" functions, introduced by Weil

$$f(\tau, \chi) = \sum_{n=1}^{\infty} a_n \chi(n) \mathrm{e}^{2\pi \mathrm{i} n \tau}$$

$$\phi_f(s, \chi) = \left(\frac{m'}{2\pi}\right)^s \Gamma(s) \sum_{n=1}^{\infty} a_n \chi(n) n^{-s} \tag{4}$$

related, respectively, to f and Φ_f. Here, χ is a primitive character modulo $m' \in \mathbb{Z}^+$, $(m', N) = 1$. Note that $\Phi_f(a, \chi)$ is the Mellin transform of $f(\tau/m', \chi)$ (see [4]).

THEOREM 1. *If $f(\tau)$ is an entire* MI *on* $\Gamma(1)$ *of weight k, with* MS υ, *and if $f(\tau\tau') = f(\tau)f(\tau')$ for all $\tau, \tau' \in H$, then $f(\tau^m)$ is an entire* MI *on* $\Gamma(1)$ *of weight mk, with* MS υ *for all $m \in \mathbb{Z}^+$ and $\tau^m \in H$.*

Proof. Let $f(\tau)$ be an entire MI on $\Gamma(1)$ of weight k, with MS υ. That is, f is holomorphic in H and satisfies equations (1). Also, f has a Fourier expansion of the form (2). Since $f(\tau\tau') = f(\tau)f(\tau')$, we obtain

$$f((\tau + 1)^m) = f^m(\tau + 1) = f^m(\tau) = f(\tau^m)$$

and

$$\tau^{-mk} f(-1/\tau^m) = \upsilon(V) f(\tau^m) + q(\tau^m).$$

Also $f(\tau^m)$ has the Fourier expansion of the form

$$f(\xi) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{2\pi \mathrm{i} n \xi}, \quad \text{for} \quad \xi = \tau^m. \tag{5}$$

$f(\tau^m)$ is holomorphic in H for $\tau^m \in H$. Hence, $f(\tau^m)$ is an entire MI on $\Gamma(1)$ of weight mk, with MS υ. □

As in [6], any RPF on $\Gamma(1)$ with poles in Q has the form

$$q(\tau^m) = \sum \alpha_1 \tau^{-ml}, \qquad -S \leq 1 \leq K. \tag{6}$$

THEOREM 2. *Let f and $f(\tau^m)$ be as in Theorem 1. The Mellin transform of the function $f(\tau^m)$ is*

$$\Phi_{(s)} = \frac{1}{m} \mathrm{i}^{\frac{s}{m}-s} (2\pi)^{-s/m} \Gamma(s/m) \sum a_n n^{-s/m}, \tag{7}$$

for $m \in \mathbb{Z}^+$, $\tau^m \in H$. We also have

$$\Phi(mk - s) = (-1)^{mk/2} \Phi(s).$$

Proof. From the Fourier expansion (5) and by the definition of Mellin transform, we have, for $\zeta = (\mathrm{i}y)^m$

$$\begin{aligned}
\phi(s) &= \int_0^\infty \left(f((\mathrm{i}y)^m) - a_0\right) y^s \frac{\mathrm{d}y}{y} \\
&= \sum_{n=1}^\infty \int_0^\infty a_n \mathrm{e}^{2\pi \mathrm{i} n \zeta} y^s \frac{\mathrm{d}y}{y} \\
&= \sum_{n=1}^\infty \int_0^\infty a_n \mathrm{e}^{-u} \left(\frac{u}{2\pi n}\right)^{s/m} \frac{1}{m} \mathrm{i}^{\frac{s}{m}-s} \frac{\mathrm{d}u}{u}.
\end{aligned}$$

Here, we used the change of variable $\mathrm{i}y = (\mathrm{i}\frac{u}{2\pi n})^{1/m}$. Thus, we obtain

$$\Phi(s) = \frac{1}{m} \mathrm{i}^{\frac{s}{m}-s} (2\pi)^{-s/m} \Gamma(s/m) \sum a_n n^{-s/m}.$$

Also, it is easy to verify that $\Phi(mk - s) = (-1)^{mk/2} \Phi(s)$. □

THEOREM 3. *Suppose f is an entire* MI *in $\Gamma(1)$. Let Φ_f be the Mellin transform of $f(\tau) - a_0$, defined by* (3). *Let Φ be the Mellin transform of $f(\tau^m) - a_0$, defined by* (7). *Then we have*

$$\Phi(s) = \frac{1}{m} \mathrm{i}^{\frac{s}{m}-s} \Phi_f(s/m).$$

Proof. This follows easily from the equations (3) and (7). □

THEOREM 4. *The Mellin transform of $f(\tau^m/m', \chi)$ is*

$$\Phi(s, \chi) = \frac{1}{m} \mathrm{i}^{\frac{s}{m}-s} (m')^{s/m} \Gamma(s/m) \sum a_n \chi(n) n^{-s/m} \tag{8}$$

for $m \in \mathbb{Z}^+$, $\tau \in H$. We also have

$$\Phi(s, \chi) = \frac{1}{m} \mathrm{i}^{\frac{s}{m}-s} \Phi_f(s/m, \chi). \tag{9}$$

Proof. (8) is proved by the change of variable $\mathrm{i}y = (\mathrm{i}\frac{um'}{2\pi n})^{1/m}$, using the Mellin transform. The functional equation (9) follows easily from the equations (4) and (8). □

In [4] Knopp proved some generalizations of Hecke's celebrated correspondence. Let $f(\tau)$ and $f(\tau^m)$ be as in Theorem 1. We now rewrite without proof some theorems from [4] for the function $f(\tau^m)$ as follows.

Theorem 5. (see [4, Theorem 1]) *Suppose that $f(\tau^m)$ is an entire* MI *on* $\Gamma(1)$ *with* RPF *of the form* (6) *for $\tau^m \in H$. Let Φ be the Mellin transform of $f(\tau^m) - a_0$, defined by* (7). *Then Φ can be continued analytically to a function meromorphic in the entire s-plane, with at most simple poles at the finite number of integer values of s/m.*

Theorem 6. (see [4, Theorem 2]) *Suppose that $f(\tau^m)$ is an entire* MI *on* $\Gamma(1)$ *for $\tau^m \in H$. Then Φ can be continued analytically to a function meromorphic in the entire s-plane, with at most simple poles at integer values of s/m. Furthermore, Φ satisfies a functional equation*

$$\Phi(mk - s) = (-1)^{mk/2}\Phi(s) + R(s),$$

where $R(s)$ is a finite complex linear combination, summed over the integers j and r, of terms of the form

$$(-1/\alpha_j)^r(B(s, r-s)\mathrm{e}^{-\pi s/2}\alpha_j^s - (-1)^{mk/2}B(mk-s, r-mk+s)\mathrm{e}^{-\pi \mathrm{i}s/2}\alpha_j^{mk-s}).$$

Here, B is the beta function and α_j are the poles of the RPF *q in the set*

$$P = \{\operatorname{Re}\tau^m > 0, \quad \operatorname{Im}\tau^m \le 0\} \cup \{\operatorname{Re}\tau^m = 0, \quad 1 \le \operatorname{Im}\tau^m < 0\}.$$

3. GENERALIZATION TO $\Gamma_*^0(N)$

Suppose that in H the function f is holomorphic and satisfies the transformation formula

$$f|T = f + q_T, \quad T \in \Gamma^0(N); \qquad f|\omega(N) = Cf + q_\omega, \tag{10}$$

where q_t, q_ω are rational functions and C is a complex number. Assume further that f has the expansion (2) at ∞. Then we call f an entire MI on $\Gamma_*^0(N)$ of weight k.

In [7] Weil developed an important generalization to $\Gamma_0^*(N)$ of the Hecke correspondence. In [4] and [5] Knopp follows the organization of [7], adapting to modular integrals on $\Gamma_0^*(N)$ the arguments that Weil developed for modular forms on $\Gamma_0^*(N)$. As in Theorem 1, suppose that f is an entire MI on $\Gamma_*^0(N)$ such that $f(\tau\tau') = f(\tau)f(\tau')$, for all $\tau, \tau' \in H$. Then $f(\tau^m)$ is an entire MI on $\Gamma_*^0(N)$ of weight mk. We now rewrite without proof some theorems from [4] and [5] for the function $f(\tau^m)$ and the group $\Gamma_*^0(N)$ as follows.

Theorem 7. (see [5]) *Suppose $f(\tau^m)$ is an entire* MI *on $\Gamma_*^0(N)$, of weight mk, for $k \in \mathbb{Z}, m \in \mathbb{Z}^+, \tau^m \in H$. Let χ be a primitive Dirichlet character modulo m', $(m', N) = 1$. Then $\Phi(s)$ and $\Phi(s, \chi)$ have meromorphic continuations to the entire s-plane, with at most simple poles at integer values of s/m lying in a left half-plane and they are bounded in every "lacunary" vertical strip*

$$\{s = \sigma + \mathrm{i}t : \sigma_1 \le \sigma \le \sigma_2, |t| \ge t_0 \ge 0\}.$$

Furthermore,

$$(-1)^{mk/2}N^{\frac{mk}{2}-s}\Phi(mk-s)=C\Phi(s)+R(s)$$

and

$$(-1)^{mk/2}(Nm'^2)^{\frac{mk}{2}-s}\Phi(mk-s,\overline{\chi})=C\frac{g(\overline{\chi})}{g(\chi)}\overline{\chi}(-N)\Phi(s,\chi)+R(s,\chi),$$

where $R(s)$ is a finite linear combination of terms having the form

$$(\sqrt{N})^{r-s}(\Gamma(r-s)\Gamma(s)(\sqrt{N}\alpha\mathrm{i})^{s-r} - C(-1)^{mk/2}\Gamma(r-mk+s)\Gamma(mk-s)(\sqrt{N}\alpha\mathrm{i})^{mk-s-r}),$$

with $r\in\mathbb{Z}^+$ and α a complex number such that $\operatorname{Im}\alpha\le 0$, and $R(a,\chi)$ is a finite linear combination of terms having the form

$$(\sqrt{N})^{r-s}m'^{r-1}g(\overline{\chi})(\overline{\chi}(Na)\Gamma(r-s)\Gamma(s)(\sqrt{N}m'\alpha\mathrm{i})^{s-r} - (-1)^{mk/2}C_\chi(-a)\Gamma(r-mk+s)(mk-s)(\sqrt{N}m'\alpha\mathrm{i})^{mk-s-r}).$$

Here, $a\in\mathbb{Z}$ is such that $1\le a\le m'$, $(a,m')=1$ and $g(\chi)$ is the Gaussian sum

$$g(\chi)=\sum_{a \bmod m'}\chi(a)\mathrm{e}^{\pi\mathrm{i}a/m'}.$$

Theorem 8. (see [4, Theorem 3]) *Let τ^m be an entire* MI *on $\Gamma^0_*(N)$ of weight $mk\le 0$, such that the* RPF*'s q_T, q_ω of* (10) *are polynomials of degree $\le -mk$, for $\tau^m\in H$. Suppose that χ is a primitive Dirichlet character modulo m' with $(m',N)=1$.*

Then $\Phi(s)$ and $\Phi(s,\chi)$ have analytic continuations to the entire s-plane with at most finitely many simple poles at nonpositive, integer values of s/m. Furthermore

$$N^{\frac{mk}{2}-s}\Phi(mk-s)=C(-1)^{mk/2}\Phi(s)$$

and

$$(Nm'^2)^{\frac{mk}{2}-s}\Phi(mk-s,\overline{\chi})=C\frac{g(\overline{\chi})}{g(\chi)}\overline{\chi}(-N)(-1)^{mk/2}\Phi(s,\chi).$$

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