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IRREGULAR FORCED ALMOST PERIODIC SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL SYSTEMS

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Abstract. Let A be an almost periodic $(n \times n)$-matrix and let φ be an almost periodic vector. Suppose that $\operatorname{mod}(A) \cap \operatorname{mod}(\varphi) = \{0\}$. We say that the almost periodic solution x of the system

$$\dot{x} = A(t)x + \varphi(t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

is irregular with respect to $\operatorname{mod}(A)$ (or partially irregular) if $(\operatorname{mod}(x)+\operatorname{mod}(\varphi)) \cap \operatorname{mod}(A) = \{0\}$, and irregular forced if at the same time $\operatorname{mod}(x) \subseteq \operatorname{mod}(\varphi)$. We prove that an irregular with respect to $\operatorname{mod}(A)$ almost periodic solution is irregular forced in non-critical and some critical cases. The necessary and sufficient conditions for existence of irregular forced solutions are obtained.

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1. INTRODUCTION

Let D be a compact subset of $\mathbb{R}^n$ and let $\mathbb{R}^{k\times m}$ be the linear space of all matrices with k rows and m columns (k and m are positive integers). By $AP(\mathbb{R}^{k\times m})$ we denote the linear space of all almost periodic functions $h : \mathbb{R} \to \mathbb{R}^{k\times m}$. By $AP(D, \mathbb{R}^{k\times m})$ we denote the linear space of all continuous functions $h : \mathbb{R} \times D \to \mathbb{R}^{k\times m}$ such that each $h \in AP(D, \mathbb{R}^{k\times m})$ is almost periodic in $t \in \mathbb{R}$ uniformly for $x \in D$. By $\operatorname{mod}(h)$ we denote a frequency module of $h \in AP(\mathbb{R}^{k\times m})$ (or $h \in AP(D, \mathbb{R}^{k\times m})$. Consider the almost periodic ordinary differential system

$$\dot{x} = f(t, x), \quad t \in \mathbb{R}, \quad x \in D, \tag{1}$$

where $f \in AP(D, \mathbb{R}^{n\times 1})$. The existence problem for almost periodic solutions to (1) is a significant problem both for qualitative theory of ordinary differential equations and for its applications to vibration theory (see [1], [19]). Many authors have investigated this problem, see e.g. [2], [3], [11], [13], [14], [16], [20]. Most of them have considered only the regular solutions x, i.e. the solutions with $\operatorname{mod}(x) = \operatorname{mod}(f)$. However, there can be various relations between $\operatorname{mod}(x)$ and $\operatorname{mod}(f)$. In [15], J. Kurzweil and O. Veivoda have obtained the necessary existence conditions for almost periodic solutions x to (1) such that $\operatorname{mod}(x) \cap \operatorname{mod}(f) = \{0\}$. We say that such solutions are irregular. The analogous problem for periodic systems was studied by H. Massera [18]. In [4], [5], [10], [12] irregular periodic, quasiperiodic, and almost periodic solutions are considered.

In [6] we have shown that some classes of quasiperiodic systems admit quasiperiodic solutions with some of the right part base frequencies. For the system (1) with

$$f(t,x) = F(t,x) + G(t,x), \quad t \in \mathbb{R}, \quad x \in D, \quad \mathrm{mod}\,(F) \cap \mathrm{mod}\,(G) = \{0\},$$

the existence criteria for almost periodic solutions x such that $(\mathrm{mod}\,(x) + \mathrm{mod}\,(G)) \cap \mathrm{mod}\,(F) = \{0\}$ are given in [7]. Such solutions are called partially irregular. In [8] we have obtained the existence conditions for almost periodic partially irregular solutions of system (1) with $f(t,x) \equiv F(t,t,x)$, where $F(t_1,t_2,x)$ is almost periodic in t_j $(j = 1,2)$ uniformly for the rest of the arguments.

In this paper we consider the linear system

$$\dot{x} = A(t)x + \varphi(t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{2}$$

where $A \in AP(\mathbb{R}^{n\times n})$, $\varphi \in AP(\mathbb{R}^{n\times 1})$, and

$$\mathrm{mod}\,(A) \cap \mathrm{mod}\,(\varphi) = \{0\}. \tag{3}$$

The almost periodic solutions x to (2) such that $\mathrm{mod}\,(x) = \mathrm{mod}\,(\varphi)$ are investigated in [4]. In [9] we have considered quasiperiodic system (2) with φ in the form of a trigonometric polynomial.

The aim of this paper is to establish the existence conditions for almost periodic irregular with respect to $\mathrm{mod}\,(A)$ solutions of linear system (2) in non-critical and some critical cases. To this end we apply the results of [18] to linear systems.

2. PRELIMINARIES

Definition 1. Let $f \in AP(D, \mathbb{R}^{n\times 1})$.

a) A real number γ is called a Fourier exponent (or frequency) of f, if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,x) \exp(-\mathrm{i}\gamma t)\mathrm{d}t \not\equiv 0 \quad \text{for } x \in D.$$

b) The set Γ of all Fourier exponents of f is called the frequency set of this function.

c) The frequency module $\mathrm{mod}\,(f)$ of f is the smallest additive group of real numbers that contains all Fourier exponents of this function.

Definition 2. Let $f \in AP(D, \mathbb{R}^{n\times 1})$ be a right side of (1) and $\mathrm{mod}\,(f)$ is splitted into direct sum of two submoduli M_1, M_2 $(j = 1,2)$, i.e. $\mathrm{mod}\,(f) = M_1 \oplus M_2$.

a) An almost periodic solution x of the system (1) is called irregular with respect to submodule M_2 (or partially irregular) if $(\mathrm{mod}\,(x) + M_1) \cap M_2 = \{0\}$.

b) An irregular with respect to submodule M_2 almost periodic solution x of the system (1) is called weakly M_2-irregular (or weakly irregular) if $\mathrm{mod}(x) \subseteq M_1$.

c) Any weakly mod (A)-irregular almost periodic solution x of the linear system (2) is called irregular forced.

In what follows we shall show a construction of transformation reducing a functional matrix to a special block form. If P is a real $(n \times n)$-matrix function defined on $\mathbb{R}$ then by $\operatorname{rank}_{\mathrm{col}} P$ we denote the column rank of P, i.e. $\operatorname{rank}_{\mathrm{col}} P$ is the maximal number of linearly independent columns of P.

LEMMA 1. *Let P be a real $(n \times n)$-matrix function defined on $\mathbb{R}$. If* $\operatorname{rank}_{\mathrm{col}} P = n - k$, $\; 0 < k < n$, *then there exists a constant nonsingular $(n \times n)$-matrix Q such that the first k columns of PQ are zero and remaining columns are linearly independent.*

Proof. If $P = (P_1, \dots, P_n)$, where $P_1, \dots, P_n$ are the columns of P, and $\operatorname{rank}_{\mathrm{col}} P < n$ then we have $a_1 P_1 + \ldots + a_n P_n = 0$ for some real numbers $a_1, \dots, a_n$ for which $a_1^2 + \ldots + a_n^2 > 0$. It follows that there exists j $(1 \leq j \leq n)$ such that $a_j \neq 0$. This yields

$$a_j P_j = -a_1 P_1 - \ldots a_{j-1} P_{j-1} - a_{j+1} P_{j+1} - \ldots - a_n P_n.$$

Take the constant $(n \times n)$-matrix S_1 which arises from the unit matrix E_n of order n by the replacement of its j-th column by the column $(a_1, \dots, a_n)^{\mathsf{T}}$. Evidently, $\det S_1 = a_j \neq 0$.

Further, we take the constant $(n \times n)$-matrix T_1 which arises from the unit matrix E_n by the exchange of its first and j-th columns. We have $\det T_1 = \pm 1 \neq 0$. Evidently, the first column of $P^{(1)} = PQ_1$ $(Q_1 = S_1 T_1)$ is zero. It is clear that $\operatorname{rank}_{\mathrm{col}} P = \operatorname{rank}_{\mathrm{col}} P^{(1)}$ because $\det Q_1 = \pm a_j \neq 0$.

If $k = 1$ then there is nothing to do. If $k > 1$ and $P_1^{(1)}, \dots, P_n^{(1)}$ are the columns of $P^{(1)}$ then with regard to $P_1^{(1)} = 0$ there exist real numbers $b_2, \dots, b_n$, not all zero, such that $b_2 P_2^{(1)} + \ldots + b_n P_n^{(1)} = 0$. We may assume that $b_r \neq 0$ for some $2 \leq r \leq n$. Then

$$b_r P_r^{(1)} = -b_2 P_2^{(1)} - \ldots - b_{r-1} P_{r-1}^{(1)} - b_{r+1} P_{r+1}^{(1)} - \ldots - b_n P_n^{(1)}.$$

Now we construct the constant $(n \times n)$-matrix S_2 which arises from E_n by the replacement of its r-th column by the column $(0, b_2, \dots, b_n)^{\mathsf{T}}$. We have $\det S_2 = b_r \neq 0$. Further, we take the constant $(n \times n)$-matrix T_2 which arises from E_n by the exchange of its second and r-th columns. Obviously, $\det T_2 \neq 0$. The first two columns of $P^{(2)} = P^{(1)} Q_2$ $(Q_2 = S_2 T_2)$ are zero. Evidently, $\operatorname{rank}_{\mathrm{col}} P = \operatorname{rank}_{\mathrm{col}} P^{(2)}$ because $\det Q_2 \neq 0$.

If $k > 2$ then we continue by the analogous way. So that we obtain the matrix $Q = S_1 T_1 \cdots S_k T_k$ such that first k columns of PQ are zero and the remaining columns are linearly independent. □

3. THEOREMS

In this section we obtain the existence conditions for almost periodic irregular with respect to mod(A) solutions of system (2). We suppose that (3) holds. Let x be an almost periodic solution to (2) and $(\operatorname{mod}(x)+\operatorname{mod}(\varphi))\cap\operatorname{mod}(A)=\{0\}$. It is clear that $x\neq 0$. By [7], the solution x satisfies the system

$$\dot{x}=\hat{A}x+\varphi(t),\quad [A(t)-\hat{A}]x=0,\quad t\in\mathbb{R},\quad \hat{A}=\lim_{T\to\infty}\frac{1}{T}\int_0^T A(t)\mathrm{d}t. \tag{4}$$

Denote $\tilde{A}=A-\hat{A}$. Since $\operatorname{mod}(x)\cap\operatorname{mod}(\tilde{A})=\{0\}$ and $x\neq 0$, it follows from $\tilde{A}(t)x=0,\ t\in\mathbb{R}$, and [7] that

$$0<\operatorname{rank}_{\mathrm{col}}\tilde{A}=r<n. \tag{5}$$

By Lemma 1 there exists a constant nonsingular $(n\times n)$-matrix Q such that the first s $(s=n-r)$ columns of $\tilde{A}Q$ are zero and the remaining r columns are linearly independent. Then substitution

$$x=Qy \tag{6}$$

reduces system (4) to the form

$$\dot{y}=By+\psi(t),\quad \tilde{B}(t)y=0,\quad t\in\mathbb{R}, \tag{7}$$

where $B=Q^{-1}\hat{A}Q$, $\psi=Q^{-1}\varphi$, and $\tilde{B}=\tilde{A}Q$. The last r columns of $\tilde{B}$ are linearly independent and from this and from $\tilde{B}y=0$ follows that the last r components of y are zero. Clearly, system (7) has almost periodic solution $y=S^{-1}x$ such that $\operatorname{mod}(y)=\operatorname{mod}(x)$. Therefore, $y=(\tilde{y}^{\mathsf{T}},0,\dots,0)^{\mathsf{T}}$, $\tilde{y}=(y_1,\dots,y_s)^{\mathsf{T}}$. Consequently, system (7) take the form

$$\begin{gathered}\dot{\tilde{y}}=B_{s,s}\tilde{y}+\psi^{(1)}(t),\quad B_{n-s,s}\tilde{y}+\psi^{(2)}(t)=0,\quad t\in\mathbb{R},\\ \tilde{y}=(y_1,\dots,y_s)^{\mathsf{T}},\quad y_{s+1}=\ldots=y_n=0,\end{gathered} \tag{8}$$

where $B_{s,s}$ and $B_{n-s,s}$ are the upper left $(s\times s)$-block and the lower left $((n-s)\times s)$-block of B respectively and $\psi^{(1)}=(\psi_1,\dots,\psi_s)^{\mathsf{T}}$; $\psi^{(2)}=(\psi_{s+1},\dots,\psi_n)^{\mathsf{T}}$, i.e.

$$B=\begin{pmatrix}B_{s,s} & B_{s,n-s}\\ B_{n-s,s} & B_{n-s,n-s}\end{pmatrix},\quad \psi=\begin{pmatrix}\psi^{(1)}\\ \psi^{(2)}\end{pmatrix}.$$

Consider the system

$$\dot{\tilde{y}}=B_{s,s}\tilde{y}+\psi^{(1)}(t),\quad t\in\mathbb{R}. \tag{9}$$

By the above, system (9) has the almost periodic solution $\tilde{y}=Q^{(-1)}_{s,n}x$, where $Q^{(-1)}_{s,n}$ is the upper $(s\times n)$-block of Q^{-1}. Since $y=Q^{-1}x$ and $y=(\tilde{y}^{\mathsf{T}},0,\dots,0)^{\mathsf{T}}$, we have $\operatorname{mod}(\tilde{y})=\operatorname{mod}(x)$. Note that $\tilde{y}$ is also a solution of the second system in (8). This implies that $\tilde{y}$ satisfies the identity

$$B_{n-s,s}\tilde{y}(t)+\psi^{(2)}(t)\equiv 0,\quad t\in\mathbb{R}. \tag{10}$$

Hence, if system (2) has an irregular with respect to $\operatorname{mod}(A)$ almost periodic solution x, then conditions (5), (10) hold, where $\tilde{y}$ is the almost periodic solution to (9) with the same frequency properties as x.

Let us show that the opposite assertion also holds. Indeed, if (5) is valid, then by Lemma 1 there exists constant nonsingular $(n \times n)$-matrix Q such that transformation (6) reduces system (2) to (7). Since the last r columns of $\tilde{B}$ are linearly independent, it follows from [7] that $y_{s+1} = \ldots = y_n = 0$. Therefore, system (7) is reduced to system (8). System (8) has the almost periodic solution $\tilde{y}$, $(\mathrm{mod}\,(\tilde{y}) + \mathrm{mod}\,(\varphi)) \cap \mathrm{mod}\,(A) = \{0\}$ by assumption. Now the identity (10) provides the existence of the almost periodic solution $\tilde{y}$ to system (8), and from (6) we get

$$x = Q(\tilde{y}^{\mathsf{T}}, 0, \ldots, 0)^{\mathsf{T}}. \tag{11}$$

It is clear that (11) is a solution to (4) and $\mathrm{mod}\,(x) = \mathrm{mod}\,(\tilde{y})$. By [7], (11) is a solution of system (2) as well.

Thus, we have proved

THEOREM 1. *Suppose that $A \in AP(\mathbb{R}^{n\times n})$ and $\varphi \in AP(\mathbb{R}^{n\times 1})$ and that* (3) *holds. The system* (2) *has the irregular with respect to* $\mathrm{mod}\,(A)$ *almost periodic solution* (11) *if and only if*

1) $\mathrm{rank_{col}}\,\tilde{A} = r \quad (0 < r < n)$;

2) *the system* (8) *has the almost periodic solution* $\tilde{y}$ *such that* $(\mathrm{mod}\,(\tilde{y}) + \mathrm{mod}\,(\varphi)) \cap \mathrm{mod}\,(A) = \{0\}$;

3) *the identity* (10) *is valid.*

Thus, the problem of existence of an almost periodic partially irregular solutions to (2) is equivalent to a similar problem for system (8).

Now assume that all eigenvalues $\lambda_1(B_{ss}), \ldots, \lambda_s(B_{ss})$ of B_{ss} have nonzero real parts

$$\mathrm{Re}\,\lambda_j(B_{ss}) \neq 0 \quad (j = \overline{1, s}). \tag{12}$$

THEOREM 2. *Suppose that the conditions* (3), (12) *hold and the system* (2) *has an almost periodic irregular with respect to* $\mathrm{mod}\,(A)$ *solution x; then this solution is irregular forced.*

Proof. Let x be an almost periodic solution of system (2) and $(\mathrm{mod}\,(x) + \mathrm{mod}\,(\varphi)) \cap \mathrm{mod}\,(A) = \{0\}$. By Theorem 1, system (9) has an almost periodic solution $\tilde{y}$ such that $\mathrm{mod}\,(\tilde{y}) = \mathrm{mod}\,(x)$. It follows from (12) and [11, p. 91] that $\mathrm{mod}\,(\tilde{y}) \subseteq \mathrm{mod}\,(\psi^{(1)})$. Since $\mathrm{mod}\,(\psi^{(1)}) \subseteq \mathrm{mod}\,(\varphi)$, we obtain $\mathrm{mod}\,(\tilde{y}) \subseteq \mathrm{mod}\,(\varphi)$. This means that $\mathrm{mod}\,(x) \subseteq \mathrm{mod}\,(\varphi)$, i.e. the solution x is irregular forced. □

It should be stressed that Theorem 2 is not valid in critical case when some of $\mathrm{Re}\,\lambda_j(B_{ss})$ is zero. However, some critical cases can be considered in similar way. Let $\lambda_j\,(B_{ss}) = \alpha_j + \mathrm{i}\beta_j$ $(\mathrm{i}^2 = -1;\ j = \overline{1, s})$. Suppose that

$$\alpha_l = 0, \quad \beta_l \in \mathrm{mod}\,(\varphi) \ \ (l = \overline{1, p};\ p \leq n), \quad \alpha_q \neq 0 \ \ (q = \overline{p+1, s});$$

$$\beta_k \neq \beta_m \ \text{ for all } \ k \neq m, \quad k, m = \{1, \ldots, p\}. \tag{13}$$

Then the conjugate system

$$\dot{z} = -B_{ss}^{\mathsf{T}} z, \quad z \in \mathbb{R}^s \tag{14}$$

has p linearly independent quasiperiodic solutions

$$z^{(1)}, \ldots, z^{(p)}, \quad \mathrm{mod}\,(z^{(j)}) \subseteq \mathrm{mod}\,(\varphi) \quad (j = \overline{1,p}). \tag{15}$$

THEOREM 3. *Suppose that* $A \in AP(\mathbb{R}^{n\times n})$, $\varphi \in AP(\mathbb{R}^{n\times 1})$ *and that* $(s\times s)$-*matrix* B_{ss} *has the eigenvalues for which* (13) *is valid.*

1) *If system* (2) *has an almost periodic irregular with respect to* $\mathrm{mod}\,(A)$ *solution* x, *then this solution is irregular forced.*

2) *System* (2) *has an irregular forced almost periodic solution if and only if* (5), (10), *and*

$$\sup_{-\infty<t<+\infty} \left| \int_{t_0}^{t} \sum_{k=1}^{s} z_k^{(j)}(\tau)\psi_k^{(j)}(\tau)\mathrm{d}\tau \right| < +\infty \quad (j = \overline{1,p}) \tag{16}$$

hold.

Proof. Let x be a partially irregular almost periodic solution of system (2). Then by Theorem 1 inequality (5) is true and system (9) has an almost periodic solution $\tilde{y}$ such that $\mathrm{mod}\,(\tilde{y}) = \mathrm{mod}\,(x)$. It follows from [17, Theorem 2] that estimate (16) holds. Note that $\tilde{y}$ satisfies (8) as well. Therefore identity (10) is valid. Since $\alpha_k = 0$, $\beta_k \in \mathrm{mod}\,(\varphi)$ $(k = \overline{1,p})$, and $\mathrm{mod}\,(\psi^{(1)}) \subseteq \mathrm{mod}\,(\varphi)$, we have $\mathrm{mod}\,(\tilde{y}) \subseteq \mathrm{mod}\,(\varphi)$. Hence, $\mathrm{mod}\,(x) \subseteq \mathrm{mod}\,(\varphi)$, i.e. the solution x is irregular forced.

Conversely, assume that the conditions of Theorem 3 hold. It follows from (5) and Lemma 1 that there exists the matrix Q such that the transformation (6) reduces system (4) to the form (7). Since the last $n-s$ columns of $\tilde{B}$ are linearly independent, we obtain $y_{s+1} = \ldots = y_n = 0$. Thus, system (7) is reduced to (8). Note that B_{ss} has the eigenvalues (13) by assumption. Consequently, (15) are the solutions to (14). By [17, Theorem 2], (13), and (16), system (9) has the almost periodic solution $\tilde{y}$ and $\mathrm{mod}\,(\tilde{y}) \subseteq \mathrm{mod}\,(\psi^{(1)}) \subseteq \mathrm{mod}\,(\varphi)$. It follows from (10) and (8) that $y = (\tilde{y}^{\mathsf{T}}, 0, \ldots, 0)^{\mathsf{T}}$ satisfies (7). By (6), we obtain the almost periodic solution $x = Qy$ of system (4). By [7], x is a solution of system (2) as well. Since $\mathrm{mod}\,(x) = \mathrm{mod}\,(\tilde{y})$, we see that the solution x is irregular forced. □

COROLLARY 1. *The system* (2) *has an almost periodic irregular with respect to* $\mathrm{mod}(A)$ *solution* x *if and only if* x *satisfies the following conditions*

$$\dot{x} = A(t_0)x + \varphi(t), \quad [A(t) - A(t_0)]x = 0, \quad t \in \mathbb{R},$$

for any $t_0 \in \mathbb{R}$.

4. AN EXAMPLE

Let a_1, a_2, and φ_1 be scalar real almost periodic nonzero functions such that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T a_j(t)\mathrm{d}t=\alpha_j \quad (j=1,2), \qquad \sup_{-\infty<t<+\infty}\left|\int_{t_0}^{t}\varphi_1(\tau)\mathrm{d}\tau\right|<+\infty.$$

Suppose that $\operatorname{mod}(a)\cap\operatorname{mod}(\varphi_1)=\{0\}$, where $a=(a_1,a_2)^{\mathsf{T}}$. Consider the system

$$\dot{x}=-a_1(t)x+a_1(t)y+\varphi_1(t),$$

$$(17)\quad \dot{y}=(1-a_1(t)-a_2(t))x+(a_1(t)+a_2(t))y+\varphi_1(t)-\varphi_2(t), \quad t,x,y\in\mathbb{R},$$

where $\varphi_2(t)=\int\limits_{t_0}^{t}\varphi_1(\tau)\mathrm{d}\tau$. It follows from [16, p. 83] that $\varphi_2(t)$ is almost periodic. Note that $\operatorname{mod}(\varphi_1)=\operatorname{mod}(\varphi_2)$. We have

$$A(t)=\begin{pmatrix}-a_1(t) & a_1(t)\\ 1-a_1(t)-a_2(t) & a_1(t)+a_2(t)\end{pmatrix},\ \hat{A}=\begin{pmatrix}-\alpha_1 & \alpha_1\\ 1-\alpha_1-\alpha_2 & \alpha_1+\alpha_2\end{pmatrix}$$

and $\operatorname{rank}_{\mathrm{col}}\tilde{A}=1<2\ \ (\tilde{A}=A-\hat{A})$. By Lemma 1 there exists a nonsingular matrix Q such that

$$\tilde{A}(t)Q=\begin{pmatrix}0 & a_1(t)-\alpha_1\\ 0 & a_1(t)-\alpha_1+a_2(t)-\alpha_2\end{pmatrix}, \quad Q=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

It follows from Theorem 1 and Theorem 3 that system (17) has an irregular forced almost periodic solution. It is easy to see that the solution $x=\varphi_2$, $y=\varphi_2$ is a solution with the required properties.

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