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DATA DEPENDENCE FOR FUNCTIONAL DIFFERENTIAL EQUATIONS OF MIXED TYPES

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Abstract. In this paper we present the data dependence and the differentiability for the solution of functional differential equations of mixed type using the techniques of Picard operators on fiber.

1. INTRODUCTION

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the following notations:

$$\begin{aligned} P(X) &:= \{Y \subset X : Y \neq \phi\} \\ I(A) &:= \{Y \in P(X) : A(Y) \subset Y\} \\ F(A) &:= \{x \in X :\ A(x) = x\} \end{aligned}$$

Definition 1.1 ([6], [8], [11]). An operator A is weakly Picard operator (WPO), if the sequence $(A^n(x))_{n\in N}$ converges, for all $x \in X$ and the limit is a fix point of A.

Definition 1.2 ([6], [11], [8]). If the operator A is WPO and $F_A = x^*$, then, by definition, A is a Picard operator (PO).

We now recall the Fiber Picard Operators Theorem that we'll use on this paper.

Theorem 1.1. *Let (X, d) a metric space and (Y, ρ) a complete metric space. Let $A : X \times Y \to X \times Y$ an operator. Assume that:*
(i) *A is triangular, $A = (B, C)$, $B : X \to X$, $C : X \times Y \to Y$;*
(ii) *A is continuous ;*
(iii) *B is a Picard operator;*
(iv) *$C(x, \cdot)$ is contraction, for all $x \in X$.*
Then the operator A is a Picard operator.

The purpose of this paper is to study the problem:

$$\begin{aligned} &(1.1) \qquad x'(t,\lambda) = f(t, x(t,\lambda), x(\phi(t),\lambda), x(\psi(t),\lambda),\lambda), \quad t \in [0,T] \\ &(1.2) \qquad x|_{[-\delta,0]} = x^0 \end{aligned}$$

where $x^0 \in C([-\delta, T])$.

2. THE EQUIVALENCE OF THE PROBLEM (1.1)+ (1.2) TO A FIXED POINT PROBLEM

We consider the equation (1.1) under the following conditions:
(i) $0 \le T \le \infty, \quad 0 \le \delta \le \infty$
(ii) $\phi \in C([0,T],[-\delta,T]), \ \phi(t) \le \ t$ on $[0,T]$
(iii) $\psi \in C([0,T],[-\delta,T]), \ \psi(t) \ge t$ on $[0,T]$
(iv) $f \in C([0,T] \times R^3 \times \Lambda), \ \Lambda \subset R, \ \Lambda$ is a compact interval
(v) f is Lipschitz and $f \in C^1$.

We say that x is a solution of the equation (1.1) if

$$x \in C([-\delta,T] \times \Lambda), \qquad x(\cdot,\lambda) \in C^1[0,T]$$

and satisfies (1.1).

Clearly, the problem (1.1) + (1.2) is equivalent with :

(2.1)
$$x(t,\lambda) = \begin{cases} x^0, & t \in [-\delta,0] \\ x^0 + \int_0^t f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)\mathrm{d}s, & t \in [0,T] \end{cases}$$

which is a fixed point problem in $C([-\delta,T] \times \Lambda)$.

Let the operator $B \ : \ C([-\delta,T] \times \Lambda) \ \to \ C([-\delta,T] \times \Lambda)$ be defined by:

(2.2)
$$B(x)(t,\lambda) = \begin{cases} x^0, & t \in [-\delta,0] \\ x^0 + \int_0^t f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)\mathrm{d}s, & t \in [0,T] \end{cases}$$

THEOREM 2.1. *Assume that* (i)–(iv) *hold. In addition suppose that*
(vi) *there exist* $\alpha,\beta,\gamma \in R_+$ *such that*

$$|f(t,x_1,y_1,z_1,\lambda) - f(t,x_2,y_2,z_2,\lambda)| \le \alpha|x_1 - x_2| + \beta|y_1 - y_2| + \gamma|z_1 - z_2|,$$

where for all $(t,x,y,z,\lambda) \in [0,T] \times R^3 \times \Lambda$

$$0 < (\alpha + \beta + \gamma \mathrm{e}^{\eta M})\eta^{-1} < 1,$$

where

$$M = \max(0, \max(\psi(t) - t)), \quad t \in [0,T].$$

Then the operator B is a Picard operator.

Proof. First we shall prove that B is a contraction. Indeed, we have

$$
\begin{aligned}
|B(x)(t,\lambda)-B(y)(t,\lambda)| \leq\ & \alpha\int_0^t |x(s,\lambda)-y(s,\lambda)|\mathrm{d}s \\
& + \beta\int_0^t |x(\phi(s),\lambda)-y(\phi(s),\lambda)|\mathrm{d}s \\
& + \gamma\int_0^t |x(\psi(s),\lambda)-y(\psi(s),\lambda)|\mathrm{d}s \\
& \leq \| x-y \|_B \int_0^t (\alpha\mathrm{e}^{\eta s}+\beta\mathrm{e}^{\eta\phi(s)}+\gamma\mathrm{e}^{\eta\psi(s)})\mathrm{d}s \\
& \leq \| x-y \|_B \int_0^t [\alpha+\beta+\gamma\mathrm{e}^{\eta(\psi(s)-s)}]\mathrm{e}^{\eta s}\mathrm{d}s \\
& \leq (\alpha+\beta+\gamma\mathrm{e}^{\eta M})\eta^{-1} \| x-y \|_B \mathrm{e}^{\eta t}
\end{aligned}
$$

It follows that

$$
\| B(x)(t,\lambda)-B(y)(t,\lambda) \|_B \leq (\alpha+\beta+\gamma\mathrm{e}^{\eta M})\eta^{-1} \| x-y \|_B .
$$

Here we denoted

$$
\|.\|_B = \max_{(t,\lambda)}(\mathrm{e}^{-\tau|t-t_0|}\|x(t,\lambda)\|_{R^n}).
$$

Thus B is a contraction, since $(\alpha+\beta+\gamma\mathrm{e}^{\eta M})\eta^{-1} < 1$. Hence B is a Picard operator. □

Remark 2.1. Under the conditions of the Theorem 2.1, the problem (1.1) + (1.2) has a unique solution in $C([-\delta,T]\times\Lambda)$.

3. MAIN RESULTS

Using a heuristic method, let us find if the solution for problem (2.1) is derivable and what the form of this solution is.

We know that the solution is continuous, and we suppose that it is also derivable. We derive the equation formally on λ.

$$
\begin{aligned}
\frac{\partial x(t,\lambda)}{\partial\lambda} =\ & \int_0^t \frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(s,\lambda)}\cdot\frac{\partial x(s,\lambda)}{\partial\lambda}\mathrm{d}s \\
& + \int_0^t \frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(\phi(s),\lambda)}\cdot\frac{\partial x(\phi(s),\lambda)}{\partial\lambda}\mathrm{d}s \\
& + \int_0^t \frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(\psi(s),\lambda)}\cdot\frac{\partial x(\psi(s),\lambda)}{\partial\lambda}\mathrm{d}s \\
& + \int_0^t \frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial\lambda}\mathrm{d}s.
\end{aligned}
$$

This relation suggests us to consider the operator

$$
C : C([-\delta,T]\times\Lambda)\times C([-\delta,T]\times\Lambda) \to C([-\delta,T]\times\Lambda)
$$

defined by

(3.1)
$$C(x,y)(t,\lambda) := \begin{cases} 0, & t\in[-\delta,0] \\ \int_0^t \frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(s,\lambda)}\cdot y(s,\lambda)\mathrm{d}s & \\ \quad +\int_0^t \frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(\phi(s),\lambda)}\cdot y(\phi(s),\lambda)\mathrm{d}s & \\ \quad +\int_0^t \frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(\psi(s),\lambda)}\cdot y(\psi(s),\lambda)\mathrm{d}s & \\ \quad +\int_0^t \frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial\lambda}\mathrm{d}s, & t\in[0,T] \end{cases}$$

First we shall prove that C is a contraction on y. Indeed, we have

$$\begin{aligned} &|C(x,y)-C(x,z)| \\ &\quad\le \int_0^t \left|\frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(s,\lambda)}[y(s,\lambda)-z(s,\lambda)]\right|\mathrm{d}s \\ &\quad+\int_0^t \left|\frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(\phi(s),\lambda)}[y(\phi(s),\lambda)-z(\phi(s),\lambda)]\right|\mathrm{d}s \\ &\quad+\int_0^t \left|\frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(\psi(s),\lambda)}[y(\psi(s),\lambda)-z(\psi(s),\lambda)]\right|\mathrm{d}s \\ &\quad\le \|x-y\|_B\Big[\int_0^t \frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(s,\lambda)}\mathrm{e}^{\eta s}\mathrm{d}s \\ &\quad+\int_0^t \frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(\phi(s),\lambda)}\mathrm{e}^{\eta s}\mathrm{d}s \\ &\quad+\int_0^t \frac{\partial f(s,x(s,\lambda),x(\phi(s),\lambda),x(\psi(s),\lambda),\lambda)}{\partial x(\psi(s),\lambda)}\mathrm{e}^{\eta s}\mathrm{d}s\Big]. \end{aligned}$$

It follows ([13])

$$\| C(x,y)(t,\lambda)-C(x,z)(t,\lambda)\|_B \le (\alpha+\beta+\gamma\mathrm{e}^{\eta M})\eta^{-1}\| x-y\|_B$$

Thus

$$C(x,\cdot): C([-\delta,T]\times\Lambda)\to C([-\delta,T]\times\Lambda)$$

is a contraction for all $x\in C([-\delta,T]\times\Lambda)$, hence $C(x,\cdot)$ is a Picard operator on $C([-\delta,T]\times\Lambda)$.

Remark 3.1. The contraction constant for the operator B is the same contraction constant for the operator C.

Consider the operator

$$A: C([-\delta,T]\times\Lambda)\times C([-\delta,T]\times\Lambda) \;\to\; C([-\delta,T]\times\Lambda)\times C([-\delta,T]\times\Lambda)$$

defined by:

$$(x,y) \longmapsto (B(x)(t,\lambda), C(x,y)(t,\lambda)).$$

By the Fiber Picard Operators Theorem the operator A is a Picard operator.

Let (x^*,y^*) be the unique fixed point of the operator A. The approximation sequence for A is

$$x_{k+1} = B(x_k), \quad x_n \rightrightarrows x^*, \quad k \in \overline{0, n-1}, \ n \in N$$
$$y_{k+1} = C(x_k, y_k), \quad y_n \rightrightarrows y^*, \quad k \in \overline{0, n-1}, \ n \in N.$$

where by " $\rightrightarrows$ " we have denoted the uniform convergence.

Let $h \in C^1([-\delta, T] \times \Lambda)$. If we choose

$$x_0 := h(t,\lambda), \qquad y_0 := \frac{\partial h(t,\lambda)}{\partial \lambda},$$

then by the induction it follows that

$$y_1 = \frac{\partial x_1}{\partial \lambda}, \dots, \ y_k = \frac{\partial x_k}{\partial \lambda}.$$

Thus

$$x_k \rightrightarrows x^*, \qquad \frac{\partial x_k}{\partial \lambda} \rightrightarrows y^*.$$

By Weierstrass' theorem it follows that

$$y^*(t,\cdot) \in C^1(\Lambda) \quad \text{and} \quad y^* = \frac{\partial x^*}{\partial \lambda}.$$

Thus we can give the following theorem.

THEOREM 3.1. *Assuming the conditions* (i) – (v) *hold, then the problem* (1.1) + (1.2) *has a unique solution and the solution is derivable on* λ.

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